Geometric Algebra for Computer Science

Answers and hints to selected drills and exercises

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When we wrote the drills and exercises for Geometric Algebra for Computer Science, we intended them to be for self-study. As such, we are tempted to release solutions to all the drills and structural exercises. However, as some instructors may wish to use these as homework questions, for now we are only releasing the solutions to all of the drills and to most of the odd numbered structural exercises. In the future, we plan to release solutions to all of the structural exercises, since solutions to these questions will likely appear on the internet anyway.

Chapter 2

Spanning Oriented Subspaces

2.12 Exercises

2.12.1 Drills

1. Compute the outer products of the following 3-space expressions, giving the results relative to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$. Show your work.

   (a) $(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_1 + \mathbf{e}_3)$
   Worked solution:

   $$\begin{aligned} &(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_1 + \mathbf{e}_3) \\ &\quad = \mathbf{e}_1 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3 \\ &\quad = -\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_3 \wedge \mathbf{e}_1 \end{aligned}$$

   (b) $(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \wedge (2\mathbf{e}_1)$
   Answer: $2\mathbf{e}_3 \wedge \mathbf{e}_1 - 2\mathbf{e}_1 \wedge \mathbf{e}_2$

   (c) $(\mathbf{e}_1 - \mathbf{e}_2) \wedge (\mathbf{e}_1 - \mathbf{e}_3)$
   Answer: $\mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2$

   (d) $(\mathbf{e}_1 + \mathbf{e}_2) \wedge (0.5\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$
   Answer: $3\mathbf{e}_2 \wedge \mathbf{e}_3 - 3\mathbf{e}_3 \wedge \mathbf{e}_1 + 1.5\,\mathbf{e}_1 \wedge \mathbf{e}_2$

   (e) $(\mathbf{e}_1 \wedge \mathbf{e}_2) \wedge (\mathbf{e}_1 + \mathbf{e}_3)$
   Answer: $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$

   (f) $(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 \wedge \mathbf{e}_3)$
   Answer: $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$

2. Given the 2-blade $\mathbf{B} = \mathbf{e}_1 \wedge (\mathbf{e}_2 - \mathbf{e}_3)$ that represents a plane, determine if each of the following vectors lies in that plane. Show your work.

   (a) $\mathbf{e}_1$
   Worked solution:

   $$\begin{aligned} \mathbf{e}_1 \wedge \mathbf{B} &= \mathbf{e}_1 \wedge (\mathbf{e}_1 \wedge (\mathbf{e}_2 - \mathbf{e}_3)) \\ &= \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge (\mathbf{e}_2 - \mathbf{e}_3) \\ &= 0 \end{aligned}$$

   Answer: In the plane.

(b) $\mathbf{e}_1 + \mathbf{e}_2$

Answer: Not in the plane.

(c) $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$

Answer: Not in the plane.

(d) $2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$

Answer: In the plane.

3. What is the area of the parallelogram spanned by the vectors $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$ and $\mathbf{b} = -\mathbf{e}_1 - \mathbf{e}_2$ (relative to the area of $\mathbf{e}_1 \wedge \mathbf{e}_2$)?

Worked solution:

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= (\mathbf{e}_1 + 2\mathbf{e}_2) \wedge (-\mathbf{e}_1 - \mathbf{e}_2) \\
&= -\mathbf{e}_1 \wedge \mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_2 - 2\mathbf{e}_2 \wedge \mathbf{e}_1 - 2\mathbf{e}_2 \wedge \mathbf{e}_2 \\
&= -\mathbf{e}_1 \wedge \mathbf{e}_2 + 2\mathbf{e}_1 \wedge \mathbf{e}_2 \\
&= \mathbf{e}_1 \wedge \mathbf{e}_2
\end{aligned}$$

Answer: 1

4. Compute the intersection of the non-homogeneous line L with support vector $\mathbf{e}_1$ and direction vector $\mathbf{e}_2$, and the line M with support vector $\mathbf{e}_2$ and direction vector $(\mathbf{e}_1 + \mathbf{e}_2)$, using 2-blades. Does the basis have to be orthonormal?

Answer: Clearly $\mathbf{x}$ should be a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. So set $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$. The demand that $\mathbf{x}$ is on L is $\mathbf{x} \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$. This yields $\alpha\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$, so that $\alpha = 1$. Similarly, the demand that $\mathbf{x}$ be on M gives $\alpha - \beta = -1$. Therefore the solution is $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$.

5. Compute $(2 + 3\mathbf{e}_3) \wedge (\mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3)$ using the grade-based defining equations of the outer product.

Worked solution:

$$\begin{aligned}
&\sum_{k=0}^{3}\sum_{\ell=0}^{3} \langle 2 + 3\mathbf{e}_3\rangle_k \wedge \langle \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3\rangle_\ell = \\
&= \langle 2 + 3\mathbf{e}_3\rangle_0 \wedge \langle \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3\rangle_1 + \\
&\quad \langle 2 + 3\mathbf{e}_3\rangle_1 \wedge \langle \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3\rangle_1 + \\
&\quad \langle 2 + 3\mathbf{e}_3\rangle_0 \wedge \langle \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3\rangle_2 + \\
&\quad \langle 2 + 3\mathbf{e}_3\rangle_1 \wedge \langle \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3\rangle_2 \\
&= 2 \wedge \mathbf{e}_1 + 3\mathbf{e}_3 \wedge \mathbf{e}_1 + 2 \wedge (\mathbf{e}_2 \wedge \mathbf{e}_3) + 3\mathbf{e}_3 \wedge (\mathbf{e}_2 \wedge \mathbf{e}_3) \\
&= 2\mathbf{e}_1 + 3\mathbf{e}_3 \wedge \mathbf{e}_1 + 2(\mathbf{e}_2 \wedge \mathbf{e}_3)
\end{aligned}$$

2.12.2 Structural Exercises

1. The outer product was defined for a vector space $\mathbb{R}^n$ without a metric, but it is of course still defined when we do have a metric space. In $\mathbb{R}^2$ with Euclidean metric, choose an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{e}_1$ is parallel to $\mathbf{a}$. Write $\mathbf{a} = \alpha\,\mathbf{e}_1$ and $\mathbf{b} = \beta\,(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2)$, where ϕ is the angle from $\mathbf{a}$ to $\mathbf{b}$. Evaluate the outer product. Your result should be:

$$\mathbf{a} \wedge \mathbf{b} = \alpha\beta\sin\phi\,(\mathbf{e}_1 \wedge \mathbf{e}_2). \tag{2.14}$$

What is the geometrical interpretation?

Answer: The outer product evaluation is straightforward. In the answer,

$\mathbf{e}_1 \wedge \mathbf{e}_2$ denotes the unit amount of area in the $\mathbf{e}_1 \wedge \mathbf{e}_2$-plane. The parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ has an $\mathbf{e}_1$-base of α, and a $\mathbf{e}_2$-height of $\beta \sin(\phi)$, so the factor $\alpha\beta \sin\phi$ is indeed the correct amount of area. The geometric algebra result gives both magnitude and attitude.

3. The anti-commutative algebra has unusual properties, so you should be careful when computing. For real numbers, $(x+y)(x-y) = x^2 - y^2$, and for the dot product of two vectors (in a metric vector space) this corresponds simply to: $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} - \mathbf{y}\cdot\mathbf{y}$. Now for comparison compute $(\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}-\mathbf{y})$ and simplify as far as possible. You should get $-2\mathbf{x} \wedge \mathbf{y}$, which is a rather different result than the other products give! Verify with a drawing that this algebraic result makes perfect sense geometrically in terms of oriented areas.

   Answer: $(\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}-\mathbf{y}) = \mathbf{x}\wedge\mathbf{x} + \mathbf{y}\wedge\mathbf{x} - \mathbf{x}\wedge\mathbf{y} - \mathbf{y}\wedge\mathbf{y} = -2\,\mathbf{x}\wedge\mathbf{y}$. The drawing involves the parallelograms formed by $\mathbf{x}\wedge\mathbf{y}$, and by $(\mathbf{x}+\mathbf{y}) \wedge (\mathbf{x}-\mathbf{y})$. You recognize some similar triangles and it is clear that the area of the latter is twice as big. But makes sure you also get the correct relative orientation of -1.

5. Consider $\mathbb{R}^4$ with basis $\{\mathbf{e}_i\}_{i=1}^4$. Show that the 2-vector $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$ is not a 2-blade. (i.e., it cannot be written as the outer product of two vectors). Hint: Set $\mathbf{a}\wedge\mathbf{b} = \mathbf{B}$, develop $\mathbf{a}$ and $\mathbf{b}$ onto the basis, expand the outer product onto the bivector basis; attempt to solve the resulting set of scalar equations.

   Answer: Set $\mathbf{a} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4$, $\mathbf{b} = a_1\,\mathbf{e}_1 + a_2\,\mathbf{e}_2 + a_3\,\mathbf{e}_3 + a_4\,\mathbf{e}_4$. Then we get $\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - b_1a_2)\mathbf{e}_1\wedge\mathbf{e}_2 + (a_3b_4 - b_3a_4)\mathbf{e}_3\wedge\mathbf{e}_4 + (a_1b_3 - b_1a_3)\mathbf{e}_1\wedge\mathbf{e}_3 + (a_1b_4 - b_1a_4)\mathbf{e}_1\wedge\mathbf{e}_4 + (a_2b_3 - b_2a_3)\mathbf{e}_2\wedge\mathbf{e}_3 + (a_2b_4 - b_2a_4)\mathbf{e}_2\wedge\mathbf{e}_4$. The first two coefficients should be 1, the rest should be 0. Those zero coefficients yield $\frac{a_1}{a_3} = \frac{b_1}{b_3}$, $\frac{a_1}{a_4} = \frac{b_1}{b_4}$, $\frac{a_2}{a_3} = \frac{b_2}{b_3}$, $\frac{a_2}{a_4} = \frac{b_2}{b_4}$. As a consequence, $\frac{a_1}{a_2} = \frac{a_1}{a_4}\frac{a_4}{a_2} = \frac{b_1}{b_4}\frac{b_4}{b_2} = \frac{b_1}{b_2}$, so that the first coefficient is also 0. That is a contradiction.

9. Prove (2.13): $\mathbf{A}_k \wedge \mathbf{B}_l = (-1)^{kl}\,\mathbf{B}_l \wedge \mathbf{A}_k$.

   Answer: This is easily shown by expanding each of the blades as an outer product of vectors, and swapping those vectors:

$$
\begin{aligned}
\mathbf{A}_k \wedge \mathbf{B}_l &= (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) \wedge (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) \\
&= \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l \\
&= (-1)\,\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge \mathbf{b}_1 \wedge \mathbf{a}_k \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l \\
&= (-1)^2\,\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{a}_k \wedge \cdots \wedge \mathbf{b}_l \\
&= \cdots \\
&= (-1)^l\,\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_{k-1} \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l \wedge \mathbf{a}_k \\
&= (-1)^{2l}\,\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_{k-2} \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l \wedge \mathbf{a}_{k-1} \wedge \mathbf{a}_k \\
&= \cdots \\
&= (-1)^{kl}\,\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_l \wedge \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k \\
&= (-1)^{kl}\mathbf{B}_l \wedge \mathbf{A}_k
\end{aligned}
$$

Chapter 3

Metric Products of Subspaces

3.11 Exercises

3.11.1 Drills

1. Let $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{b} = \mathbf{e}_2 + \mathbf{e}_3$ in a 3-dimensional Euclidean space $\mathbb{R}^{3,0}$ with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Compute the following expressions, giving the results relative to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$. Show your work.

   (a) $\mathbf{e}_1 \rfloor \mathbf{a}$
   Worked solution:

$$\begin{aligned}
\mathbf{e}_1 \rfloor \mathbf{a} &= \mathbf{e}_1 \rfloor (\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{e}_1 \rfloor \mathbf{e}_1 + \mathbf{e}_1 \rfloor \mathbf{e}_2 \\
&= 1 + 0 = 1
\end{aligned}$$

   Answer: 1

   (b) $\mathbf{e}_1 \rfloor (\mathbf{a} \wedge \mathbf{b})$
   Worked solution:

$$\begin{aligned}
\mathbf{e}_1 \rfloor (\mathbf{a} \wedge \mathbf{b}) &= \mathbf{e}_1 \rfloor \big((\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_2 + \mathbf{e}_3)\big) \\
&= \mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3) \\
&= \mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2) + \mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_3) + \mathbf{e}_1 \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_3) \\
&= (\mathbf{e}_1 \rfloor \mathbf{e}_1) \wedge \mathbf{e}_2 - \mathbf{e}_1 \wedge (\mathbf{e}_1 \rfloor \mathbf{e}_2) + (\mathbf{e}_1 \rfloor \mathbf{e}_1) \wedge \mathbf{e}_3 - \mathbf{e}_1 \wedge (\mathbf{e}_1 \rfloor \mathbf{e}_3) + \\
&\quad (\mathbf{e}_1 \rfloor \mathbf{e}_2) \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge (\mathbf{e}_1 \wedge \mathbf{e}_3) \\
&= \mathbf{e}_2 - 0 + \mathbf{e}_3 - 0 + 0 - 0
\end{aligned}$$

   Alternatively, develop the contraction on the bivector components:

$$\begin{aligned}
\mathbf{e}_1 \rfloor (\mathbf{a} \wedge \mathbf{b}) &= \mathbf{e}_1 \rfloor ((\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_2 + \mathbf{e}_3)) \\
&= \big(\mathbf{e}_1 \rfloor (\mathbf{e}_1 + \mathbf{e}_2)\big) \wedge (\mathbf{e}_2 + \mathbf{e}_3)) - (\mathbf{e}_1 + \mathbf{e}_2) \wedge \big(\mathbf{e}_1 \rfloor (\mathbf{e}_2 + \mathbf{e}_3)\big) \\
&= (\mathbf{e}_2 + \mathbf{e}_3) + 0
\end{aligned}$$

   Answer: $\mathbf{e}_2 + \mathbf{e}_3$

(c) $(\mathbf{a} \wedge \mathbf{b}) \rfloor \mathbf{e}_1$

Answer: 0

(d) $(2\mathbf{a} + \mathbf{b}) \rfloor (\mathbf{a} + \mathbf{b})$

Answer: 9

(e) $\mathbf{a} \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$

Answer: $\mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_1$

(f) $\mathbf{a}^*$

Answer: $-\mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_3 \wedge \mathbf{e}_1$

(g) $(\mathbf{a} \wedge \mathbf{b})^*$

Answer: $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$

(h) $\mathbf{a} \rfloor \mathbf{b}^*$

Answer: $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$

2. Compute the cosine of the angle between the following subspaces given on an orthonormal basis of a Euclidean space:

(a) $\mathbf{e}_1$ and $\alpha\,\mathbf{e}_1$

Answer: 1

(b) $(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3$ and $\mathbf{e}_1 \wedge \mathbf{e}_3$

Answer: $1/\sqrt{2}$

(c) $(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) \wedge \mathbf{e}_3$ and $\mathbf{e}_2 \wedge \mathbf{e}_3$

Answer: $\sin\phi$

(d) $\mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{e}_3 \wedge \mathbf{e}_4$

Answer: 0

3. Set up and draw the reciprocal frame for vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, on an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ represented as $\mathbf{b}_1 = \mathbf{e}_1$ and $\mathbf{b}_2 = \mathbf{e}_1 + \mathbf{e}_2$. Use the reciprocal frame to compute the coordinates of the vector $\mathbf{x} = 3\mathbf{e}_1 + \mathbf{e}_2$ on the $\{\mathbf{b}_1, \mathbf{b}_2\}$-basis.

Partial Worked Solution:

$$
\begin{aligned}
\mathbf{I}_2 &= \mathbf{b}_1 \wedge \mathbf{b}_2 \\
\mathbf{I}_2^{-1} &= \mathbf{b}_2 \wedge \mathbf{b}_1 \\
\mathbf{b}^1 &= -1^0 \mathbf{b}_2 \rfloor \mathbf{I}_2^{-1} \\
&= \mathbf{b}_2 \rfloor (\mathbf{b}_2 \wedge \mathbf{b}_1) \\
&= (\mathbf{b}_2 \rfloor \mathbf{b}_2) \wedge \mathbf{b}_1 + -1^1 \mathbf{b}_2 \wedge (\mathbf{b}_2 \mathbf{b}_1) \\
&= ((\mathbf{e}_1 + \mathbf{e}_2) \rfloor (\mathbf{e}_1 + \mathbf{e}_2)) \wedge \mathbf{b}_1 - \mathbf{b}_2 \wedge ((\mathbf{e}_1 + \mathbf{e}_2) \rfloor \mathbf{e}_1) \\
&= 2\mathbf{b}_1 - \mathbf{b}_2 = \mathbf{e}_1 - \mathbf{e}_2
\end{aligned}
$$

Answer: $\mathbf{b}^1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b}^2 = \mathbf{e}_2$, and $\mathbf{x} = 2\mathbf{b}_1 + \mathbf{b}_2$.

3.11.2 Structural exercises

1. In 2-dimensional Euclidean space $\mathbb{R}^{2,0}$ with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, let us determine the value of the contraction $\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)$ by means of its implicit definition (3.6) with $\mathbf{A} = \mathbf{e}_1$ and $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$. Let $\mathbf{X}$ range over the basis of the blades: $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$. This produces four equations, each of which gives you information on the coefficient of the corresponding basis element in

the final result. Show that $\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2) = 0\,(1) + 0\,(\mathbf{e}_1) + 1\,(\mathbf{e}_2) + 0\,(\mathbf{e}_1 \wedge \mathbf{e}_2)$.

Answer: Use (3.6) as $\mathbf{X} * (\mathbf{e}_1 \rfloor \mathbf{e}_1 \wedge \mathbf{e}_2)) = (\mathbf{X} \wedge \mathbf{e}_1) * (\mathbf{e}_1 \wedge \mathbf{e}_2)$:

$$\begin{aligned}
1 * (\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)) &= (1 \wedge \mathbf{e}_1) * (\mathbf{e}_1 \wedge \mathbf{e}_2) = 0 \\
\mathbf{e}_1 * (\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)) &= (\mathbf{e}_1 \wedge \mathbf{e}_1) * (\mathbf{e}_1 \wedge \mathbf{e}_2) = 0 \\
\mathbf{e}_2 * (\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)) &= (\mathbf{e}_2 \wedge \mathbf{e}_1) * (\mathbf{e}_1 \wedge \mathbf{e}_2) = 1 \\
(\mathbf{e}_1 \wedge \mathbf{e}_2) * (\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)) &= (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1) * (\mathbf{e}_1 \wedge \mathbf{e}_2) = 0.
\end{aligned}$$

Therefore $\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)$ has only an $\mathbf{e}_2$ component, which equals 1, so $\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_2$.

3. Derive the following dualities for the right contraction, corresponding to (3.20) and (3.21) for the usual (left) contraction:

$$\begin{aligned}
\mathbf{C} \lfloor (\mathbf{B} \wedge \mathbf{A}) &= (\mathbf{C} \lfloor \mathbf{B}) \lfloor \mathbf{A} \quad \text{universally valid} && (3.33) \\
\mathbf{C} \lfloor (\mathbf{B} \lfloor \mathbf{A}) &= (\mathbf{C} \lfloor \mathbf{B}) \wedge \mathbf{A} \quad \text{when } \mathbf{A} \subseteq \mathbf{C} && (3.34)
\end{aligned}$$

Then give the counterpart of (3.24). (Hint: use (3.19).)

Answer: Using (3.19), these can be converted to proven statements about the left contraction. For the first statement, this gives:

$$\mathbf{C} \lfloor (\mathbf{B} \wedge \mathbf{A}) = \big((\widetilde{\mathbf{A}} \wedge \widetilde{\mathbf{B}}) \rfloor \widetilde{\mathbf{C}}\big)^{\sim} = \big(\widetilde{\mathbf{A}} \rfloor (\widetilde{\mathbf{B}} \rfloor \widetilde{\mathbf{C}})\big)^{\sim} = (\widetilde{\mathbf{B}} \rfloor \widetilde{\mathbf{C}})^{\sim} \lfloor \mathbf{A} = (\mathbf{C} \lfloor \mathbf{B}) \lfloor \mathbf{A}.$$

The second statement is similar.

5. The equation $\mathbf{x} \rfloor \alpha = 0$ (in (3.8)) also has a consistent geometric interpretation in the sense of Section 3.3. Since the scalar α denotes the point at the origin, $\mathbf{x} \rfloor \alpha$ has the following semantics: 'the subspace of vectors perpendicular to $\mathbf{x}$, contained in the 0-blade α'. Give a plausible correctness argument of this statement.

Answer: The only 'vector' contained in α is 0, so that is the result. Since the answer should be a blade of grade $0 - 1 = -1$, combining this result with the recursion formula (B.4) in Appendix B explains why blades of negative grade should be algebraically 0. Geometrically, this means that they do not exist.

7. Duality in 1-dimensional space should avoid the extra sign involved in double duality, as specified in (3.24). Show this explicitly, by taking the dual of a vector $\mathbf{a}$ relative to a suitably chosen unit pseudoscalar for the space, and dualizing again.

Answer: All vectors are proportional to some unit vector $\mathbf{e}$, so set $\mathbf{a} = \alpha\mathbf{e}$. Then $\mathbf{a}^* = \alpha\mathbf{e} \rfloor \mathbf{e}^{-1} = \alpha$. Double dualization gives $(\mathbf{a}^*)^* = \alpha\mathbf{e}^{-1} = \alpha\mathbf{e}/\mathbf{e}^2 = \mathbf{a}/\mathbf{e}^2$. In a Euclidean space, this in indeed equal to $\mathbf{a}$, but in an anti-Euclidean space where $\mathbf{e}^2 = -1$, it is not. The problem (and (3.24)) needs this refinement.

9. In a plane with unit pseudoscalar $\mathbf{I}_2$, we can rotate a vector by a straight angle using the contraction: $\mathbf{x} \rfloor \mathbf{I}_2$ is a perpendicular to $\mathbf{x}$. Therefore you can construct an orthogonal basis for the plane from any vector in it. Use this capability to give a coordinate-free specification of a rotation of a vector $\mathbf{x}$, over ϕ radians in that plane. Make sure you get the rotation direction correctly related to the plane's orientation. (We will do rotations properly in Chapter 7.)

Answer: $R[\mathbf{x}] = \cos(\phi)\mathbf{x} - \sin(\phi)\mathbf{x} \rfloor \mathbf{I}_2$.

11. Derive the notorious '*bac-cab* formula' for the cross product (i.e., $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$), directly from its definition (3.28). What is the corresponding formula using $\wedge$ and $\rfloor$, and its geometric interpretation?

Answer:

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})^*)^* \\ &= -\mathbf{a} \rfloor (\mathbf{b} \wedge \mathbf{c}) \\ &= \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}).\end{aligned}$$

This is essentially the formula for the contraction product of a vector onto a 2-blade, a special instance of the much more general 'passing through' formula of (3.16). The geometric semantics of both formulations is clearly the same. It constructs a vector that resides in the plane of the 2-blade, perpendicular to the vector $\mathbf{a}$. This derivation is so straightforward that you can now forget the formula.

13. In a non-orthonormal basis, the outer product $\mathbf{b}^i \wedge \mathbf{b}_i$ of a vector and its corresponding reciprocal is not generally zero. However, when summed over all basis vectors, all those 2-blades cancel out:

$$\sum_i \mathbf{b}^i \wedge \mathbf{b}_i = 0. \tag{3.35}$$

Show this by expressing $\mathbf{b}^i$ on the usual basis $\{\mathbf{b}_j\}$, and using a symmetry argument on the resulting double summation.

Answer: To avoid trivial signs, let us compute the reverse order:

$$\begin{aligned}&\sum_i \mathbf{b}_i \wedge \mathbf{b}^i = \\ &\quad= \sum_i (-1)^{i-1}\, \mathbf{b}_i \wedge (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \mathbf{b}_n)/\mathbf{I}_n \\ &\quad= \big(\sum_i (-1)^{i-1}\, \mathbf{b}_i \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \mathbf{b}_n)\big)/\mathbf{I}_n \\ &\quad= \sum_i (-1)^{i-1} \sum_{j<i} (-1)^{j-1} (\mathbf{b}_i \cdot \mathbf{b}_j)\,(\mathbf{b}_1 \wedge \cdots \wedge \breve{\mathbf{b}}_j \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \mathbf{b}_n)/\mathbf{I}_n \\ &\qquad+ \sum_i (-1)^{i-1} \sum_{j>i} (-1)^{j} (\mathbf{b}_i \cdot \mathbf{b}_j)\,(\mathbf{b}_1 \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \breve{\mathbf{b}}_j \wedge \cdots \wedge \mathbf{b}_n)/\mathbf{I}_n \\ &\quad= \sum_i \sum_{j<i} (-1)^{i+j}\,(\mathbf{b}_i \cdot \mathbf{b}_j)\,(\mathbf{b}_1 \wedge \cdots \wedge \breve{\mathbf{b}}_j \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \mathbf{b}_n)/\mathbf{I}_n \\ &\qquad- \sum_i \sum_{j>i} (-1)^{i+j}\,(\mathbf{b}_i \cdot \mathbf{b}_j)\,(\mathbf{b}_1 \wedge \cdots \wedge \breve{\mathbf{b}}_i \wedge \cdots \wedge \breve{\mathbf{b}}_j \wedge \cdots \wedge \mathbf{b}_n)/\mathbf{I}_n.\end{aligned}$$

The sign is different depending on whether j is less than or more than i. But the terms look the same under interchange of j and i. When doing the double sum, each such term involving $(\mathbf{b}_i \cdot \mathbf{b}_j)$ or $(\mathbf{b}_j \cdot \mathbf{b}_i)$ occurs twice, with two opposite signs. Therefore the total result is zero.

Chapter 4

Linear Transformations of Subspaces

4.9 Structural Exercises

1. Point mirroring in 3D space leads to a change of orientation of the volume 3-blades. We know this 'spatial inversion' better from reflection in a mirror. Show that that has indeed the same effect. (Hint: Let the mirror plane be characterized by a 2-blade $\mathbf{B}$, and let $\mathbf{a}$ be a vector perpendicular to $\mathbf{B}$ (for example, $\mathbf{a} = \mathbf{B}^*$). Then define the linear transformation performing the mirror reflection, and apply it to a sensibly chosen 3-blade in this set-up. Why does your result generalize to arbitrary 3-blades?)

   Answer: The reflection in the $\mathbf{B}$-plane, characterized by $\mathbf{a} = \mathbf{B}^*$, would be $\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{a})\,\mathbf{a}^{-1}$. Obviously, under this reflection, $\mathbf{a}$ becomes $-\mathbf{a}$, but any vector in $\mathbf{B}$ is perpendicular to $\mathbf{a}$ and remains unchanged. Therefore a 3-blade $\mathbf{a} \wedge \mathbf{B}$ becomes $-\mathbf{a} \wedge \mathbf{B}$. Since all 3-blades are proportional to each other in 3D, and the reflection is linear, all 3-blades change to minus themselves.

3. You may want to apply a linear mapping f to a k-dimensional subspace. You could then be tempted to use (4.7) with its pseudoscalar $\mathbf{I}_k$ substituted for $\mathbf{I}_n$, to define what the determinant of f is on this subspace. Why doesn't this work?

   Answer: The linear mapping may not have its results purely in the subspace $\mathbf{I}_k$, and then the division does not produce a scalar that can be used as determinant.

5. Design a non-trivial linear map $\mathsf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ which has an eigenvector and an eigen-2-blade, both with eigenvalue 1.

   Answer: On a not necessarily orthonormal basis, define the map by $\mathsf{f}[\mathbf{e}_1] = \mathbf{e}_1$ and $\mathsf{f}[\mathbf{e}_2] = \mathbf{e}_2$.

7. To continue with the previous problem after you know about the adjoint in Section 4.3.2, rewrite the correct expression for the squared norm of $\mathsf{f}[\mathbf{A}]$ in the form $\mathbf{A} * \mathsf{g}[\mathbf{A}]$, and determine g in terms of f. This is the *metric mapping* corresponding to the transformation f, and it shows that the transformed space can be treated as a space with a new inner product $\mathbf{a} \cdot \mathbf{b} \equiv \mathbf{a} * \mathsf{g}[\mathbf{b}]$.

   Answer: The true squared norm is $\mathsf{f}[\mathbf{A}] * \mathsf{f}[\mathbf{A}]^{\sim} = \mathsf{f}[\mathbf{A}] * \mathsf{f}[\widetilde{\mathbf{A}}] = \mathbf{A} * \overline{\mathsf{f}}[\mathsf{f}[\widetilde{\mathbf{A}}]] = (\overline{\mathsf{f}} \circ \mathsf{f})[\widetilde{\mathbf{A}}]$, involving the composition of the linear transformations $\overline{\mathsf{f}}$ and f, which we write compactly as $\overline{\mathsf{f}}\mathsf{f}$. As the exercise is phrased, the mapping g is thus $\bar{\mathsf{f}}\mathsf{f}$ preceded (or followed) by a reversion. It is actually more common to define g through the squared norm being $\mathbf{A} * \mathsf{g}[\widetilde{\mathbf{A}}]$. Then $\mathsf{g} = \bar{\mathsf{f}}\mathsf{f}$.

9. Show that in a space $\mathbb{R}^n$ with arbitrary basis $\{\mathbf{b}_i\}_{i=1}^n$, the adjoint of a linear transformation f can be constructed as

$$\bar{\mathsf{f}}[\mathbf{x}] = \sum_{i=1}^{n} (\mathbf{x} * \mathsf{f}[\mathbf{b}_i])\, \mathbf{b}^i. \tag{4.19}$$

Answer: We need to show that this is the adjoint, by showing that it satisfies the implicit definition (4.10). We use linearity a few times:

$$\begin{aligned}
\bar{\mathsf{f}}[\mathbf{x}] * \mathbf{a} &= \sum_{i=1}^{n} (\mathbf{x} * \mathsf{f}[\mathbf{b}_i])\,(\mathbf{b}^i * \mathbf{a}) \\
&= \mathbf{x} * \Big(\sum_i \mathsf{f}[\mathbf{b}_i]\,(\mathbf{b}^i * \mathbf{a})\Big) \\
&= \mathbf{x} * \mathsf{f}[\sum_i \mathbf{b}_i\,(\mathbf{b}^i * \mathbf{a})] \\
&= \mathbf{x} * \mathsf{f}[\mathbf{a}],
\end{aligned}$$

where the last transition recognizes that the sum is just the expansion of $\mathbf{a}$ on a complete basis for $\mathbb{R}^n$. So $\bar{\mathsf{f}}$ as defined has the fundamental property (4.10).

11. Give an expression for $\bar{\mathsf{f}}[\mathbf{A} \rfloor \mathbf{B}]$. Hint: consider the symmetry of (4.10).
Answer: $\bar{\mathsf{f}}[\mathbf{A} \rfloor \mathbf{B}] = \mathsf{f}^{-1}[\mathbf{A}] \rfloor \bar{\mathsf{f}}[\mathbf{B}]$.

13. For the shear $\mathbf{x} \mapsto \mathsf{f}_s[\mathbf{x}] \equiv \mathbf{x} + s\,(\mathbf{x} \cdot \mathbf{e}_1)\,\mathbf{e}_1$ (on the standard orthonormal basis of $\mathbb{R}^{n,0}$) compute the transformation matrix $[\![\mathsf{f}_s]\!]$ (to act on vectors). Also compute the matrix $[\![\mathsf{f}_s^*]\!]$. Verify the results in a picture of the shear of a planar line and its normal vector.

Answer: **Erratum**:*This is* non-uniform scaling *rather than* shear, *which would be* $\mathbf{x} \mapsto \mathsf{f}_s[\mathbf{x}] \equiv \mathbf{x} + s\,(\mathbf{x} \cdot \mathbf{e}_2)\,\mathbf{e}_1$. *Let us treat both.*
To compute the matrix for f, element (i, j) is made as $\mathbf{e}_i \rfloor \mathsf{f}[\mathbf{e}_j]$. That gives the following matrix: $[\![\mathsf{f}]\!] = \left[\!\!\left[\begin{matrix} 1+s & 0 \\ 0 & 1 \end{matrix}\right]\!\!\right]$. For instance, the (1,1) entry is

$$\mathbf{e}_1 \rfloor \mathsf{f}[\mathbf{e}_1] = \mathbf{e}_1 \rfloor ((1+s)\mathbf{e}_1) = \mathbf{e}_1.$$

For f^*, we have to compute things like

$$\begin{aligned}
\mathbf{e}_1 \rfloor \mathsf{f}^*[\mathbf{e}_1] &= \mathbf{e}_1 \rfloor (\mathsf{f}[{\mathbf{e}_1}^{-*}]^*) \\
&= \mathbf{e}_1 \rfloor (\mathsf{f}[\mathbf{e}_1 \rfloor (\mathbf{e}_1 \wedge \mathbf{e}_2)] \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \\
&= \mathbf{e}_1 \rfloor (\mathsf{f}[\mathbf{e}_2] \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \\
&= \mathbf{e}_1 \rfloor (\mathbf{e}_2 \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \\
&= \mathbf{e}_1 \rfloor \mathbf{e}_1) = 1.
\end{aligned}$$

This yields the matrix $[\![\mathsf{f}^*]\!] = \left[\!\!\left[\begin{matrix} 1 & 0 \\ 0 & 1+s \end{matrix}\right]\!\!\right]$, and the mapping $\mathbf{x} \mapsto \mathbf{x} + s(\mathbf{x} \cdot \mathbf{e}_2)\mathbf{e}_2$.

For a line through the origin with direction vector $\mathbf{u}$ we get the result $\mathsf{f}[\mathbf{u}] = \mathbf{u} + s(\mathbf{u} \cdot \mathbf{e}_1)\,\mathbf{e}_1$, and for its normal vector we would get $\mathsf{f}^*[\mathbf{n}] = \mathbf{u} + s(\mathbf{u} \cdot \mathbf{e}_2)\,\mathbf{e}_2$.

Since $\mathbf{u}\cdot\mathbf{n}=0$, we find $\mathsf{f}[\mathbf{u}]\cdot\mathsf{f}^*[\mathbf{n}]=\mathbf{u}\cdot\mathbf{n}+s(\mathbf{u}\cdot\mathbf{e}_1)\,(\mathbf{e}_1\cdot\mathbf{n})+s(\mathbf{n}\cdot\mathbf{e}_2)\,(\mathbf{e}_2\cdot\mathbf{u})=(1+s)\mathbf{u}\cdot\mathbf{n}=0$. So indeed the normal vector transforms correctly by f^*.

Had we taken the actual shear $\mathsf{g}[\mathbf{x}]\equiv\mathbf{x}+s\,(\mathbf{x}\cdot\mathbf{e}_2)\,\mathbf{e}_1$, we would find $\mathsf{g}^*[\mathbf{x}]\equiv\mathbf{x}-s\,(\mathbf{x}\cdot\mathbf{e}_1)\,\mathbf{e}_2$, $[\![\mathsf{g}]\!]=\left[\!\!\left[\begin{matrix}1 & s\\ 0 & 1\end{matrix}\right]\!\!\right]$ and $[\![\mathsf{g}^*]\!]=\left[\!\!\left[\begin{matrix}1 & 0\\ -s & 1\end{matrix}\right]\!\!\right]$. Again, line $\mathbf{u}$ and normal $\mathbf{n}$ remain orthogonal: $\mathsf{g}[\mathbf{u}]\cdot\mathsf{g}^*[\mathbf{n}]=\mathbf{u}\cdot\mathbf{n}+s\,(\mathbf{u}\cdot\mathbf{e}_2)\,(\mathbf{e}_1\cdot\mathbf{n})-s\,(\mathbf{u}\cdot\mathbf{e}_2)\,(\mathbf{e}_1\cdot\mathbf{n})=\mathbf{u}\cdot\mathbf{n}=0$.

15. The classical closed-form formula for the inverse of a matrix $[\![\mathsf{A}]\!]$ is

$$[\![\mathsf{A}]\!]^{-1}=\frac{\mathrm{adj}([\![\mathsf{A}]\!])}{\det([\![\mathsf{A}]\!])},\tag{4.20}$$

where $\mathrm{adj}([\![\mathsf{A}]\!])$ is the classical 'adjoint matrix', of which the (i,j)th element equals $(-1)^{i+j}\det([\![\mathsf{A}_{ji}]\!])$, with $[\![\mathsf{A}_{ji}]\!]$ a 'minor matrix' obtained from $[\![\mathsf{A}]\!]$ by omitting the j-th row and the i-th column. Show that this terrific coordinate-based construction is identical to the coordinate-free formula (4.16). Equation (4.20) is very hard to compute with algebraically, though we will say that it is easy to implement. (Though in practice, one implements matrix inversion by Gaussian elimination, so that eq.(**??**) is usually treated as little more than a mathematical curiosity, neither good for derivation nor for implementation.)

Answer: Comparison of this formula with our inverse in (4.16) shows that we should be able to demonstrate

$$\mathrm{adj}([\![\mathsf{A}]\!])[\![\mathbf{x}]\!]=[\![\mathsf{A}[\mathbf{x}^*]^{-*}]\!].$$

It is a pain to do this symbolically, leading to a swamp of indices. Rather, let us give how it works in 3D. In that case

$$[\![\mathsf{A}]\!]=\left[\!\!\left[\begin{matrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{matrix}\right]\!\!\right],$$

and the (2,1)-entry of $\mathrm{adj}([\![\mathsf{A}]\!])$ is then minus the determinant of the matrix without the first row and second column (note the transposition involved!), which yields $a_{31}a_{23}-a_{21}a_{33}$. The same element would be computed in geometric algebra as the $\mathbf{e}_1$-component of the transformation on $\mathbf{e}_2$, which is

$$\begin{aligned}
\mathbf{e}_1\rfloor\mathsf{A}[\mathbf{e}_2{}^*]^{-*} &= \mathbf{e}_1\rfloor\mathsf{A}[\mathbf{e}_2\rfloor(\mathbf{e}_3\wedge\mathbf{e}_2\wedge\mathbf{e}_1)]^{-*}\\
&= \mathbf{e}_1\rfloor\mathsf{A}[\mathbf{e}_1\wedge\mathbf{e}_3]^{-*}\\
&= \mathbf{e}_1\rfloor(\mathsf{A}[\mathbf{e}_1]\wedge\mathsf{A}[\mathbf{e}_3])^{-*}\\
&= \big(\mathbf{e}_1\wedge\mathsf{A}[\mathbf{e}_1]\wedge\mathsf{A}[\mathbf{e}_3]\big)^{-*}\\
&= \big(\mathbf{e}_1\wedge(a_{11}\mathbf{e}_1+a_{21}\mathbf{e}_2+a_{31}\mathbf{e}_3)\wedge(a_{13}\mathbf{e}_1+a_{23}\mathbf{e}_2+a_{33}\mathbf{e}_3)\big)^{-*}\\
&= \big(\mathbf{e}_1\wedge(a_{21}a_{33}-a_{31}a_{23})\,\mathbf{e}_2\wedge\mathbf{e}_3\big)^{-*}\\
&= (a_{21}a_{33}-a_{31}a_{23})(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3)\rfloor(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3)\\
&= a_{31}a_{23}-a_{21}a_{33}.
\end{aligned}$$

So the answer is the same. You see how the presence of the $\mathbf{e}_2$ in this example effectively performs the elimination of the second column through its outer product; and how the fact that we contraction with $\mathbf{e}_1$ selects all but the first row. The antisymmetry of the outer product takes care of all the signs. That pattern generalizes in a straightforward manner. *By the way, what we called the 'minor matrix' is also known as the 'cofactor matrix'.*

17. In standard linear algebra, one way to encode a subspace is as the image of a matrix. The subspace spanned by the basis $\{\mathbf{b}_1, \cdots, \mathbf{b}_k\}$ is then the image of the matrix $[\![\mathbf{B}]\!] = [\![\mathbf{b}_1 \cdots \mathbf{b}_k]\!]$. The orthogonal projection of a vector $\mathbf{x}$ onto this subspace im$[\![\mathbf{B}]\!]$ is computed using the *projection matrix* as the vector

$$[\![\mathbf{B}]\!]([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])^{-1}[\![\mathbf{B}]\!]^T[\![\mathbf{x}]\!].$$

Show that this is in fact the same mapping as our $(\mathbf{x}\rfloor\mathbf{B})\rfloor\mathbf{B}^{-1}$ of (3.25). How would you describe the extension as an outermorphism in standard linear algebra?

Answer: We rewrite the geometric algebra expression slightly differently to make it more recognizable. Let k be the grade of $\mathbf{B}$. We develop using (3.16), and then using (3.31) we get:

$$\begin{aligned}
(\mathbf{x}\rfloor\mathbf{B})\rfloor\mathbf{B}^{-1} &= \big(\sum_{i=1}^{k}(-1)^{i-1}\mathbf{b}_1\wedge\cdots\wedge(\mathbf{x}\rfloor\mathbf{b}_i)\wedge\cdots\mathbf{b}_k\big)\rfloor\mathbf{B}^{-1}\\
&= \big(\sum_{i=1}^{k}(\mathbf{x}\cdot\mathbf{b}_i)\,(-1)^{i-1}\mathbf{b}_1\wedge\cdots\wedge\breve{\mathbf{b}}_i\wedge\cdots\mathbf{b}_k\big)\rfloor\mathbf{B}^{-1}\\
&= \sum_{i=1}^{k}(\mathbf{x}\rfloor\mathbf{b}_i)\,\mathbf{b}^i.
\end{aligned}$$

So the projection is just a composition of $\mathbf{x}$ using the vectors in $\mathbf{B}$; it uses the reciprocal frame since those vectors need not be orthonormal.

The matrix expression $[\![\mathbf{B}]\!]([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])^{-1}[\![\mathbf{B}]\!]^T[\![\mathbf{x}]\!]$ contains the inner product of the vectors of $\mathbf{B}$ with $\mathbf{x}$ in its last term $[\![\mathbf{B}]\!]^T[\![\mathbf{x}]\!]$. The two expressions are therefore equivalent if the first part defines the matrix $[\![\mathbf{B}']\!]$ of the reciprocal frame $[\![\mathbf{b}^1\ \mathbf{b}^2\cdots\mathbf{b}^k]\!]$. That demand is equivalent to wanting $[\![\mathbf{B}]\!]^T\,[\![\mathbf{B}']\!]$ to be the identity, since $\mathbf{b}_i\cdot\mathbf{b}^j = \delta_i^j$. And indeed it is:

$$[\![\mathbf{B}]\!]^T\,[\![\mathbf{B}']\!] = [\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!]\,([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])^{-1} = \big([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!]\big)\,([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])^{-1} == [\![1]\!]_{k\times k}.$$

Therefore the two expressions are identical.

It is instructive to work this out for $k=2$; a large number of terms appears in both approaches, constructed in different manners, but ultimately identical. First let us compute the geometric algebra expression:

$$\begin{aligned}
&(\mathbf{x}\rfloor(\mathbf{b}_1\wedge\mathbf{b}_2))\rfloor(\mathbf{b}_1\wedge\mathbf{b}_2)^{-1} =\\
&\quad= \frac{\big((\mathbf{x}\cdot\mathbf{b}_1)\,\mathbf{b}_2-(\mathbf{x}\cdot\mathbf{b}_2)\,\mathbf{b}_1\big)\rfloor(\mathbf{b}_2\wedge\mathbf{b}_1)}{(\mathbf{b}_1\wedge\mathbf{b}_2)*(\mathbf{b}_2\wedge\mathbf{b}_1)}\\
&\quad= \frac{\big((\mathbf{x}\cdot\mathbf{b}_1)\,(\mathbf{b}_2\cdot\mathbf{b}_2)-(\mathbf{x}\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_2)\big)\mathbf{b}_1+\big(-(\mathbf{x}\cdot\mathbf{b}_1)\,(\mathbf{b}_2\cdot\mathbf{b}_1)+(\mathbf{x}\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_1)\big)\mathbf{b}_2}{(\mathbf{b}_2\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_1)-(\mathbf{b}_1\cdot\mathbf{b}_2)^2}
\end{aligned}$$

Now work out the matrix expression. The hardest part there is the inverse:

$$\begin{aligned}
([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])^{-1} &= \left(\left[\!\!\left[\begin{matrix}\mathbf{b}_1{}^T\\ \mathbf{b}_2{}^T\end{matrix}\right]\!\!\right][\![\begin{matrix}\mathbf{b}_1 & \mathbf{b}_2\end{matrix}]\!]\right)^{-1}\\
&= \left[\!\!\left[\begin{matrix}\mathbf{b}_1\cdot\mathbf{b}_1 & \mathbf{b}_1\cdot\mathbf{b}_2\\ \mathbf{b}_1\cdot\mathbf{b}_2 & \mathbf{b}_2\cdot\mathbf{b}_2\end{matrix}\right]\!\!\right]^{-1}\\
&= \left[\!\!\left[\begin{matrix}\mathbf{b}_2\cdot\mathbf{b}_2 & -\mathbf{b}_1\cdot\mathbf{b}_2\\ -\mathbf{b}_1\cdot\mathbf{b}_2 & \mathbf{b}_1\cdot\mathbf{b}_1\end{matrix}\right]\!\!\right]\Big/\big((\mathbf{b}_2\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_1)-(\mathbf{b}_1\cdot\mathbf{b}_2)^2\big)
\end{aligned}$$

Multiplying by $[\![\mathbf{B}]\!]$ on the left then gives what our reasoning exposed as the reciprocal frame matrix:

$$[\![\mathbf{B}']\!] = \left[\!\!\left[\frac{(\mathbf{b}_2\cdot\mathbf{b}_2)\mathbf{b}_1 - (\mathbf{b}_1\cdot\mathbf{b}_2)\mathbf{b}_2}{(\mathbf{b}_2\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_1) - (\mathbf{b}_1\cdot\mathbf{b}_2)^2} \quad \frac{-(\mathbf{b}_1\cdot\mathbf{b}_2)\mathbf{b}_1 + (\mathbf{b}_1\cdot\mathbf{b}_1)\mathbf{b}_2}{(\mathbf{b}_2\cdot\mathbf{b}_2)(\mathbf{b}_1\cdot\mathbf{b}_1) - (\mathbf{b}_1\cdot\mathbf{b}_2)^2} \right]\!\!\right],$$

and subsequent right multiplication by $[\![\mathbf{B}]\!][\![\mathbf{x}]\!]$ indeed yields the above result, showing equivalence in this grade-2 case.

By the way, note that the determinant computation required for the matrix inverse $\det([\![\mathbf{B}]\!]^T[\![\mathbf{B}]\!])$ is precisely the quantity $\|\mathbf{B}\| = \mathbf{B} * \widetilde{\mathbf{B}}$ required in the inverse of the blade $\mathbf{B}$. In fact, it matches the definition of the scalar product (3.2) literally.

As to the second part of the question, extending to an outermorphism would allow the projection matrix to be applied to a subspace $\mathbf{X}$ spanned by vectors $\mathbf{x}_1, \cdots, \mathbf{x}_m$. Such a subspace would be encoded by the matrix $[\![\mathbf{X}]\!] = [\![\mathbf{x}_1 \cdots \mathbf{x}_m]\!]$, and the projection matrix can be applied to that immediately, to produce the transformed subspace matrix. Some care is required, though: since we are doing a projection, the result may be in a smaller-dimensional subspace, so the columns of the result are possibly dependent. If one wants a minimal representation, dependent columns need to be eliminated, in a procedure that makes the total operation non-linear. The geometric algebra result in such a case would be zero (for instance, projecting a 3-blade to a 2-blade).

Chapter 5

Intersection and Union of Subspaces

5.10 Exercises

5.10.1 Drills

Compute join $\mathbf{A} \cup \mathbf{B}$ and meet $\mathbf{A} \cap \mathbf{B}$ for the following blades:

1. $\mathbf{A} = \mathbf{e}_1$ and $\mathbf{B} = \mathbf{e}_2$.

   Worked answer: The join will be proportional to the pseudo-scalar $\mathbf{I}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$. So letting $\mathbf{A} \cup \mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$ we have

$$\begin{aligned} \mathbf{A} \cap \mathbf{B} &= (\mathbf{B} \rfloor \mathbf{I}_2^{-1}) \rfloor \mathbf{A} \\ &= (\mathbf{e}_2 \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \rfloor \mathbf{e}_1 \\ &= \Big((\mathbf{e}_2 \rfloor \mathbf{e}_2) \wedge \mathbf{e}_1 + (-1)^1 \mathbf{e}_2 \wedge (\mathbf{e}_2 \rfloor \mathbf{e}_1) \Big) \rfloor \mathbf{e}_1 \\ &= (1 \wedge \mathbf{e}_1 + 0) \rfloor \mathbf{e}_1 \\ &= 1. \end{aligned}$$

2. $\mathbf{A} = \mathbf{e}_2$ and $\mathbf{B} = \mathbf{e}_1$.

   Answer: Again the join can be $\mathbf{e}_1 \wedge \mathbf{e}_2$. Now compute the meet as $(\mathbf{e}_1 \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \rfloor \mathbf{e}_2 = -\mathbf{e}_1 \rfloor \mathbf{e}_1 = -1$.

3. $\mathbf{A} = \mathbf{e}_1$ and $\mathbf{B} = 2\mathbf{e}_1$.

   Answer: The join can be $\mathbf{e}_1$. Now compute the meet as $(2\mathbf{e}_1 \rfloor \mathbf{e}_1) \rfloor \mathbf{e}_1 = 2\mathbf{e}_1$.

4. $\mathbf{A} = \mathbf{e}_1$ and $\mathbf{B} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$.

   Answer: The join can be $\mathbf{e}_1 \wedge \mathbf{e}_2$. Now compute the meet as $((\mathbf{e}_1 + \mathbf{e}_2) \rfloor (\mathbf{e}_2 \wedge \mathbf{e}_1)) \rfloor \mathbf{e}_1/\sqrt{2} = 1/\sqrt{2}$.

5. $\mathbf{A} = \mathbf{e}_1$ and $\mathbf{B} = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2$.

   Answer: join is $\mathbf{e}_1 \wedge \mathbf{e}_2$, meet is $\sin\phi$.

6. $\mathbf{A} = \mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{B} = \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2$.

   Answer: join is $\mathbf{A}$, meet is $\mathbf{B}$.

7. $\mathbf{A} = \mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{B} = \mathbf{e}_2$.

   Answer: join is $\mathbf{e}_1 \wedge \mathbf{e}_2$, meet is $\mathbf{e}_2$.

8. $\mathbf{A} = \mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{B} = \mathbf{e}_2 + 0.00001\,\mathbf{e}_3$.

   Answer: join is $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$, meet is 0.00001. A small change compared to the previous drill leads to a large difference in outcome!

5.10.2 Structural Exercises

1. There is an interesting reciprocal relationship between **A**, **B**, **J** and **M**.

$$(\mathbf{B}\rfloor\mathbf{J}^{-1}) * (\mathbf{A}\lfloor\mathbf{M}^{-1}) = 1$$

   Verify the steps in the following proof: $1 = \mathbf{M}^{-1} * \mathbf{M} = \mathbf{M}^{-1} * ((\mathbf{B}\rfloor\mathbf{J}^{-1})\rfloor\mathbf{A}) = (\mathbf{M}^{-1} \wedge (\mathbf{B}\rfloor\mathbf{J}^{-1})) * \mathbf{A} = (\mathbf{B}\rfloor\mathbf{J}^{-1}) * (\mathbf{A}\lfloor\mathbf{M}^{-1})$. Then prove in similar manner:

$$(\mathbf{M}^{-1}\rfloor\mathbf{B}) * (\mathbf{J}^{-1}\lfloor\mathbf{A}) = 1$$

   Answer:

$$\begin{aligned} 1 &= \mathbf{M}^{-1} * \mathbf{M} \\ &= \mathbf{M}^{-1} * ((\mathbf{B}\rfloor\mathbf{J}^{-1})\rfloor\mathbf{A}) && (5.6) \\ &= (\mathbf{M}^{-1} \wedge (\mathbf{B}\rfloor\mathbf{J}^{-1})) * \mathbf{A} && (3.6) \\ &= (\mathbf{B}\rfloor\mathbf{J}^{-1}) * (\mathbf{A}\lfloor\mathbf{M}^{-1}) && (3.18) \end{aligned}$$

   $1 = \mathbf{J}^{-1} * \mathbf{J} = \mathbf{J}^{-1} * (\mathbf{A} \wedge (\mathbf{M}^{-1}\rfloor\mathbf{B})) = (\mathbf{J}^{-1}\lfloor\mathbf{A}) * (\mathbf{M}^{-1}\rfloor\mathbf{B})$.

3. Compute meet and join of two vectors **a** and **b** in general position, and show that the magnitude (relative to their join) is the sine of their angle. Relate the sign of the sine to the order of intersection. In this case the meet should be anti-symmetric.

   Answer: Let $\mathbf{a} = \mathbf{e}_1$ and $\mathbf{b} = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2$, where θ is the angle from **a** to **b**. Let $\mathbf{J} = \mathbf{e}_1 \wedge \mathbf{e}_2$. Then $\mathbf{J}^{-1} = -\mathbf{e}_1 \wedge \mathbf{e}_2$ and

$$\begin{aligned} \mathbf{a} \cap \mathbf{b} &= (\mathbf{b}\rfloor\mathbf{J}^{-1})\rfloor\mathbf{a} \\ &= -((\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2)\rfloor(\mathbf{e}_1 \wedge \mathbf{e}_2))\rfloor\mathbf{e}_1 \\ &= -(\cos(\theta)\mathbf{e}_1\rfloor(\mathbf{e}_1 \wedge \mathbf{e}_2))\rfloor\mathbf{e}_1 - (\sin(\theta)\mathbf{e}_2\rfloor(\mathbf{e}_1 \wedge \mathbf{e}_2))\rfloor\mathbf{e}_1 \\ &= -\cos(\theta)((\mathbf{e}_1\rfloor\mathbf{e}_1) \wedge \mathbf{e}_2 - \mathbf{e}_1 \wedge (\mathbf{e}_1\rfloor\mathbf{e}_2))\rfloor\mathbf{e}_1 \\ &\quad - \sin(\theta)((\mathbf{e}_2\rfloor\mathbf{e}_1) \wedge \mathbf{e}_2 - \mathbf{e}_1 \wedge (\mathbf{e}_2\rfloor\mathbf{e}_2))\rfloor\mathbf{e}_1 \\ &= (\sin(\theta)\mathbf{e}_1 - \cos(\theta)\mathbf{e}_2)\rfloor\mathbf{e}_1 \\ &= \sin(\theta) \end{aligned}$$

   Since the answer is the sine of the angle θ between **a** and **b**, it changes sign when we swap them. This is more clearly visible when we dare to use a more symbolic computation. We know that the join is the common plane **I**, so $\mathbf{b}^*$ is a vector; then use symmetry of the inner product, and duality:

$$\mathbf{a} \cap \mathbf{b} = \mathbf{b}^* \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b}^* = (\mathbf{a} \wedge \mathbf{b})^*.$$

   The weight of the bivector $\mathbf{a} \wedge \mathbf{b}$ relative to the unit pseudoscalar of the plane is $\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin(\theta)$ (by (2.14)), so the result follows for the unit vectors. Antisymmetry of the meet is now obvious from the antisymmetry of $\mathbf{a} \wedge \mathbf{b}$.

5. As an exercise in symbolic manipulation of the products so far, let us consider the meet of $\mathbf{a} \wedge \mathbf{B}$ and $\mathbf{a} \wedge \mathbf{C}$, where **a** is a vector and the blades **B** and **C** have no common factor. The answer should obviously be proportional to **a**, but what precisely is the proportionality factor? (Hint: If you get stuck, the next exercise derives the answer as $(\mathbf{a} \wedge \mathbf{B} \wedge \mathbf{C})^*$.)

7. Use the previous derivation to derive the general *factorization of the* meet:

$$(\mathbf{A} \wedge \mathbf{B}) \cap (\mathbf{A} \wedge \mathbf{C}) = \mathbf{A}\,(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})^*, \tag{5.11}$$

   where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have no common factors.

   Answer: The derivation follows the same pattern. Note that the final result is a multiple of $\mathbf{A}$, since $(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C})^*$ is just the scalar weight of the common span relative to the unit pseudoscalar of the join.

Chapter 6

The Fundamental Product of Geometric Algebra

6.7 Exercises

6.7.1 Drills

1. Let $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{b} = \mathbf{e}_2 + \mathbf{e}_3$ in a 3-dimensional Euclidean space with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Compute the following expressions, giving the results relative to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$. Show your work.

   (a) $\mathbf{a}\,\mathbf{a}$
   Worked answer:

$$\begin{aligned}
\mathbf{a}\,\mathbf{a} &= \mathbf{a} \rfloor \mathbf{a} + \mathbf{a} \wedge \mathbf{a} \\
&= (\mathbf{e}_1 + \mathbf{e}_2) \rfloor (\mathbf{e}_1 + \mathbf{e}_2) \\
&= \mathbf{e}_1 \rfloor \mathbf{e}_1 + \mathbf{e}_1 \rfloor \mathbf{e}_2 + \mathbf{e}_2 \rfloor \mathbf{e}_1 + \mathbf{e}_2 \rfloor \mathbf{e}_2 \\
&= 1 + 0 + 0 + 1 = 2
\end{aligned}$$

   or alternatively

$$\begin{aligned}
\mathbf{a}\,\mathbf{a} &= (\mathbf{e}_1 + \mathbf{e}_2)\,(\mathbf{e}_1 + \mathbf{e}_2) \\
&= (\mathbf{e}_1\,\mathbf{e}_1) + (\mathbf{e}_1\,\mathbf{e}_2) + (\mathbf{e}_2\,\mathbf{e}_1) + (\mathbf{e}_2\,\mathbf{e}_2) \\
&= (\mathbf{e}_1\,\mathbf{e}_1) + 2(\mathbf{e}_1 \cdot \mathbf{e}_2) + (\mathbf{e}_2\,\mathbf{e}_2) \\
&= 1 + 0 + 1 = 2
\end{aligned}$$

   Answer: 2

   (b) $\mathbf{a}\,\mathbf{b}$
   Answer: $1 + \mathbf{e}_2 \wedge \mathbf{e}_3 - \mathbf{e}_3 \wedge \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2$

   (c) $\mathbf{b}\,\mathbf{a}$
   Answer: $1 - \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_2$

   (d) $(\mathbf{e}_1 \wedge \mathbf{e}_2)\,\mathbf{a}$
   Worked answer:

$$\begin{aligned}
(\mathbf{e}_1 \wedge \mathbf{e}_2)\,\mathbf{a} &= (\mathbf{e}_1 \wedge \mathbf{e}_2)\,(\mathbf{e}_1 + \mathbf{e}_2) \\
&= (\mathbf{e}_1 \wedge \mathbf{e}_2)\,\mathbf{e}_1 + (\mathbf{e}_1 \wedge \mathbf{e}_2)\,\mathbf{e}_2
\end{aligned}$$

$$\begin{aligned} &= (\mathbf{e}_1\,\mathbf{e}_2)\,\mathbf{e}_1 + (\mathbf{e}_1\,\mathbf{e}_2)\,\mathbf{e}_2 \\ &= (-\mathbf{e}_2\,\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{e}_1\,\mathbf{e}_2)\,\mathbf{e}_2 \\ &= -\mathbf{e}_2\,(\mathbf{e}_1\,\mathbf{e}_1) + \mathbf{e}_1\,(\mathbf{e}_2\,\mathbf{e}_2) \\ &= -\mathbf{e}_2 + \mathbf{e}_1 \end{aligned}$$

From now on, you should be able to do such computations as one-liners, using anticommutativity and associativity without spelling them out, just swapping indices to bring terms together, and keeping track of the signs:

$$(\mathbf{e}_1 \wedge \mathbf{e}_2)\,(\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{e}_{12}\,\mathbf{e}_1 + \mathbf{e}_{12}\,\mathbf{e}_2 = -\mathbf{e}_{21}\,\mathbf{e}_1 + \mathbf{e}_1 = -\mathbf{e}_2 + \mathbf{e}_1$$

(e) $\mathbf{a}\,(\mathbf{e}_1 \wedge \mathbf{e}_2)$

Answer: $-\mathbf{e}_1 + \mathbf{e}_2$

(f) $(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)\,\mathbf{a}$

Answer: $\mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_3 \wedge \mathbf{e}_1$

(g) $\mathbf{a}^{-1}$

Answer: $\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$

(h) $\mathbf{b}\,\mathbf{a}^{-1}$

Answer: $\frac{1}{2}(1 - \mathbf{e}_{23} + \mathbf{e}_{31} - \mathbf{e}_{12})$

(i) $(\mathbf{e}_1 \wedge \mathbf{e}_2)^{-1}$

Answer: $-\mathbf{e}_1 \wedge \mathbf{e}_2$

2. Make a full geometric product multiplication table for the 8 basis elements $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1, \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\}$; (a) in a Euclidean metric $\mathbb{R}^{3,0}$ and (b) in a metric $\mathbb{R}^{2,1}$ with $\mathbf{e}_1 \cdot \mathbf{e}_1 = -1$.

Answer:

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_1$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$-\mathbf{e}_2$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_3$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	-1

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{123}$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{123}$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$\mathbf{e}_{123}$	$-\mathbf{e}_3$	$\mathbf{e}_2$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_1$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$-\mathbf{e}_3$	$\mathbf{e}_{123}$	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_2$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_{123}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$\mathbf{e}_3$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$	$-\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	1

6.7.2 Structural Exercises

1. Section 6.1.1 demonstrated the non-invertibility of contraction and outer product. Show by a geometrical example that the cross product of two vectors is

not invertible either. Also give an algebraic argument based on its (invertible) relationship to the outer product.

Answer: $\mathbf{x} \times \mathbf{a}$ has the same value as $(\mathbf{x} + \lambda\mathbf{a}) \times \mathbf{a}$, since the cross product is anti-symmetric. And of course the cross product is related to the outer product by $\mathbf{x} \times \mathbf{a} = (\mathbf{x} \wedge \mathbf{a})^*$. This relationship is invertible, so any properties of the outer product are similar to those of the cross product, including invertibility.

3. The outer product can be defined as the completely anti-symmetric summed average of all permutations of geometric products of its factors, with a sign for each term depending on oddness or evenness of the permutation. For the 3-blade, this means:

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{3!}\,(\mathbf{x}\,\mathbf{y}\,\mathbf{z} - \mathbf{y}\,\mathbf{x}\,\mathbf{z} + \mathbf{y}\,\mathbf{z}\,\mathbf{x} - \mathbf{z}\,\mathbf{y}\,\mathbf{x} + \mathbf{z}\,\mathbf{x}\,\mathbf{y} - \mathbf{x}\,\mathbf{z}\,\mathbf{y})$$

Derive this formula.

Answer: You might try $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{2}\,\mathbf{x} \wedge (\mathbf{y}\,\mathbf{z} - \mathbf{z}\,\mathbf{y}) = \frac{1}{4}\,(\mathbf{x}\,\mathbf{y}\,\mathbf{z} - \mathbf{x}\,\mathbf{z}\,\mathbf{y} - \mathbf{y}\,\mathbf{z}\,\mathbf{x} + \mathbf{z}\,\mathbf{y}\,\mathbf{x})$. Also true, but not what was asked. Make the starting point more symmetrical by writing $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{6}(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} - \mathbf{y} \wedge \mathbf{x} \wedge \mathbf{z} + \mathbf{y} \wedge \mathbf{z} \wedge \mathbf{x} - \mathbf{z} \wedge \mathbf{y} \wedge \mathbf{x} + \mathbf{z} \wedge \mathbf{x} \wedge \mathbf{y} - \mathbf{x} \wedge \mathbf{z} \wedge \mathbf{y})$, then perform the same trick. Terms should now group and cancel, to produce the result.

5. Show that the definition of the scalar product as $\mathbf{A} * \mathbf{B} = \langle \mathbf{A}\,\mathbf{B} \rangle_0$ is equivalent to the determinant definition of (3.2). You will then also understand why the matrix in the latter definition has the apparently reversed $\mathbf{a}_i \cdot \mathbf{b}_{k-j}$ as element (i, j) for k-blades.

Answer: To prevent a difficult administrative notation, we just show the pattern of the Laplace expansion of a determinant appearing:

$$\begin{aligned}
\langle \mathbf{A}\,\mathbf{B} \rangle_0 &= \langle (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k)\,(\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k) \rangle_0 \\
&= (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \rfloor (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k) \\
&= (\mathbf{a}_k \cdot \mathbf{b}_1)\,(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1}) \rfloor (\mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) \\
&\quad -(\mathbf{a}_k \cdot \mathbf{b}_2)\,(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1}) \rfloor (\mathbf{b}_1 \wedge \mathbf{b}_3 \wedge \cdots \wedge \mathbf{b}_k) \\
&\quad +(\mathbf{a}_k \cdot \mathbf{b}_3)\,(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1}) \rfloor (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_3 \cdots \wedge \mathbf{b}_k) \\
&\quad - \cdots \\
&\quad +(-1)^{k-1}(\mathbf{a}_k \cdot \mathbf{b}_k)\,(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_{k-1}) \rfloor (\mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_{k-1})
\end{aligned}$$

The smaller contractions can be expanded again, and the familiar anti-symmetric pattern of the determinant appears. It is clear why $\mathbf{a}_k$ combines with $\mathbf{b}_1$ for the first positive term: in the geometric product they are right next to each other. So in writing the scalar product as a determinant of a matrix, $\mathbf{a}_1 \cdot \mathbf{b}_k$ can be chosen as the (1,1)-element, and the rest follows from the patterns.

7. In the formula $(\mathbf{x} \rfloor \mathbf{A}^{-1})\,\mathbf{A}$, we can replace the geometric product by a contraction, so that it is in fact the projection $(\mathbf{x} \rfloor \mathbf{A}^{-1}) \rfloor \mathbf{A}$. Show this, using the suggestion that $\mathbf{x} \rfloor \mathbf{A}^{-1}$ might be a sub-blade of $\mathbf{A}$ — which you first need to demonstrate. After that, decompose $\mathbf{x} \rfloor \mathbf{A}^{-1}$ as a product of orthogonal vectors, and evaluate the two formulas.

Answer: First, $\mathbf{x} \rfloor \mathbf{A}^{-1}$ differs only by the scalar $1/\|\mathbf{A}\|^2$ from $\mathbf{x} \rfloor \mathbf{A}$, so it is definitely contained in $\mathbf{A}$ and is a blade. Now perform an orthogonal factorization of the blade $\mathbf{x} \rfloor \mathbf{A}^{-1}$ as $\mathbf{a}_{k-1} \cdots \mathbf{a}_2\,\mathbf{a}_k$, where all the factors are orthogonal, and use the same factorization for $\mathbf{A}$, with one more factor $\mathbf{a}_k$, so that $\mathbf{A} = \mathbf{a}_1\,\mathbf{a}_2 \cdots \mathbf{a}_k$. Now using these factorizations, compute both

$(\mathbf{x}\rfloor\mathbf{A}^{-1})\rfloor\mathbf{A}$ and $(\mathbf{x}\rfloor\mathbf{A}^{-1})\,\mathbf{A}$. Due to orthogonality of the factors, you find $(\mathbf{a}_1\cdot\mathbf{a}_1)\cdots(\mathbf{a}_{k-1}\cdot\mathbf{a}_{k-1})\,\mathbf{a}_k$ and $\mathbf{a}_1^2\cdots\mathbf{a}_{k-1}^2\,\mathbf{a}_k$, respectively. Those are of course identical.

9. In a 4-dimensional space with orthonormal basis $\{\mathbf{e}_i\}_{i=1}^4$, project the 2-blade $\mathbf{X} = (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 + \mathbf{e}_4)$ onto the 2-blade $\mathbf{A} = (\mathbf{e}_1 \wedge \mathbf{e}_3)$. Then determine the rejection as the difference of $\mathbf{X}$ and its projection. Show that this is not a blade. (See also structural exercise **??** of Chapter 2.)

   Answer: Writing $\mathbf{X}$ out in the basis 2-blades, it is $\mathbf{X} = \mathbf{e}_{13} + \mathbf{e}_{23} + \mathbf{e}_{24} + \mathbf{e}_{14}$. The projection onto $\mathbf{e}_{13}$ is simply computed as $\mathbf{e}_{13}$. The multivector $\mathbf{Y} \equiv \mathbf{X} - \mathbf{e}_{13} = \mathbf{e}_{23} + \mathbf{e}_{24} + \mathbf{e}_{14}$ has no vectors, for when you try to solve $\mathbf{x} \wedge \mathbf{Y} = 0$ for the coefficients ξ_i of $\mathbf{x}$, only $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$ results. Therefore $\mathbf{Y}$ is not a blade.

Chapter 7

Orthogonal Transformations as Versors

7.10 Exercises

7.10.1 Drills

1. Compute $R_1 \equiv R_{\mathbf{e}_1 \wedge \mathbf{e}_2\, \pi/2}$, and apply to $\mathbf{e}_1$

   Answer: Check our notation, the subscript gives $\mathbf{I}\phi$ but the rotor should be computed as $\exp(-\mathbf{I}\phi/2)$. We get $R_1 = (1 - \mathbf{e}_1\mathbf{e}_2)/\sqrt{2}$, and

$$
\begin{aligned}
R_1 \mathbf{e}_1 \widetilde{R}_1 &= \tfrac{1}{2}(1 - \mathbf{e}_1\mathbf{e}_2)\, \mathbf{e}_1 (1 + \mathbf{e}_1\mathbf{e}_2) \\
&= \tfrac{1}{2}(\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2) \\
&= \tfrac{1}{2}(\mathbf{e}_1 + 2\mathbf{e}_1^2\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_1^2\mathbf{e}_2^2) = \mathbf{e}_2.
\end{aligned}
$$

2. Compute $R_2 \equiv \exp(\mathbf{e}_3 \wedge \mathbf{e}_1\, \pi/4)$, and apply to $\mathbf{e}_2 \wedge \mathbf{e}_4$

   Answer: $R_2 = (1 + \mathbf{e}_3\mathbf{e}_1)/\sqrt{2}$, and $R_2(\mathbf{e}_1\mathbf{e}_4)\widetilde{R}_2 = (\mathbf{e}_1\mathbf{e}_4)R_2\widetilde{R}_2 = \mathbf{e}_1\mathbf{e}_4$ is invariant, since it commutes with the rotor.

3. Compute $R_2\, R_1$, and apply to $\mathbf{e}_1 \wedge \mathbf{e}_2$.

   Answer: $R_2\, R_1 = \frac{1}{2}(1 - \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1)$. This turns $\mathbf{e}_1 \wedge \mathbf{e}_2$ into $-\mathbf{e}_2 \wedge \mathbf{e}_3$.

4. Compute the axis and angle of $R_2\, R_1$.

   Answer: We have $R_2\, R_1 = \frac{1}{2}(1 - \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1)\frac{1}{2} + \frac{1}{2}\sqrt{3}\,(-\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1)/\sqrt{3}$. Comparing to $\cos(\phi/2) - \mathbf{I}\sin(\phi/2) = \cos(\phi/2) + \mathbf{a}\mathbf{I}_3^{-1}\sin(\phi/2)$, we read off that $\cos(\phi/2) = \frac{1}{2}$, $\sin(\phi/2) = \frac{1}{2}\sqrt{3}$, so that $\phi = 2\pi/3$, and $\mathbf{a} = (-\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1)/\sqrt{3}\,\mathbf{I}_3 = (\mathbf{e}_3 - \mathbf{e}_1 - \mathbf{e}_2)/\sqrt{3}$.

5. Compute the product of the rotors $R_{\mathbf{e}_{14}\pi/2}$ and $R_{\mathbf{e}_{23}\pi/2}$, and apply to $\mathbf{e}_{12}$.

   Answer: Ambiguously formulated. Let us take the reading order as the order of performing them (so that the total rotor is R_2R_1), then $R_1 = (1 - \mathbf{e}_1\mathbf{e}_4)/\sqrt{2}$ and $R_2 = (1 - \mathbf{e}_2\mathbf{e}_3)/\sqrt{2}$ and $R_2\, R_1 = \frac{1}{2}(1 - \mathbf{e}_2\,\mathbf{e}_3 - \mathbf{e}_1\mathbf{e}_4 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4)$. Applying this to $\mathbf{e}_{12}$ yields $-\mathbf{e}_{34}$.

6. Reflect $(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3$ in the plane $\mathbf{e}_1 \wedge \mathbf{e}_4$.

   Answer: $(-1)^{2(2+1)}\mathbf{e}_1\mathbf{e}_4((\mathbf{e}_1 + \mathbf{e}_2)\mathbf{e}_3)\mathbf{e}_4\mathbf{e}_1 = (-\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3$.

7. Reflect the dual plane reflector $\mathbf{e}_1$ in the plane $\mathbf{e}_1 \wedge \mathbf{e}_3$.

   Answer: $(-1)^{1(2+1)}\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_1$, it is invariant.

7.10.2 Structural Exercises

1. The generalization of the line reflection from $\mathbf{a}\,\mathbf{x}\mathbf{a}^{-1}$ to $\mathbf{a}\,\mathbf{X}\,\mathbf{a}^{-1}$ seems straightforward when we remember that a k-blade can be written as the geometric product of k mutually orthogonal vectors: $\mathbf{X} = \mathbf{x}_1\,\mathbf{x}_2\cdots\mathbf{x}_k$, and then simply compute the outermorphism as: $(\mathbf{a}\,\mathbf{x}_1\mathbf{a}^{-1})\,(\mathbf{a}\,\mathbf{x}_2\mathbf{a}^{-1})\cdots(\mathbf{a}\,\mathbf{x}_k\mathbf{a}^{-1}) = \mathbf{a}\,\mathbf{X}\mathbf{a}^{-1}$. The result is correct but the proof is wrong as it stands. Why? (Hint: Can you guarantee the factorization after reflection?)

3. Verify that a line reflection in 3-D can be performed as a rotation. Which rotation? Give the axis and angle. Verify that this reflection can be applied to any blade.

   Answer: The formula $\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1}$ can be written in terms of a 2-blade as versor using $\mathbf{x}\,\mathbf{I}_3 = \mathbf{I}_3\,\mathbf{x}$. Then $\mathbf{a}\,\mathbf{x}\,\mathbf{a}^{-1} = \mathbf{a}\,\mathbf{I}_3^{-1}\,\mathbf{I}_3\,\mathbf{x}\,\mathbf{a}^{-1} = \mathbf{a}\,\mathbf{I}_3^{-1}\,\mathbf{x}\,\mathbf{I}_3\,\mathbf{a}^{-1} = (\mathbf{a}\,\mathbf{I}_3^{-1})\,\mathbf{x}\,(\mathbf{a}\,\mathbf{I}_3)^{-1}$. Let us take $\mathbf{a}$ as unit vector, then this is the correct formula for a rotation, with versor $\mathbf{a}\,\mathbf{I}_3^{-1} = \mathbf{a}^{-*} = \exp(\mathbf{a}^{-*}\pi/2)$. The rotation axis is $\mathbf{a}$, the rotation angle is π. Since the formula is in versor form, it extends in a structure preserving manner to all elements.

5. Show from the definition of the adjoint (in Section 4.3.2) that the adjoint of a transformation that can be written as a versor product with a versor V is a versor product with the versor V^{-1}. Relate this to the orthogonality of a versor-based transformation.

   Answer: $(V\,\mathbf{x}\,V^{-1}) * \mathbf{y} = \langle V\,\mathbf{x}\,V^{-1}\,\mathbf{y}\rangle_0 = \langle \mathbf{x}\,V^{-1}\,\mathbf{y}\,V\rangle_0 = \mathbf{x} * (V^{-1}\,\mathbf{y}\,V)$, so the adjoint is characterized by V. For an orthogonal transformation like versor sandwiching, the adjoint should indeed equal the inverse.

7. Match the computation of the composition of 2-D rotations in Section 7.3.1 to that of the 3-D rotations in Section 7.3.3, both algebraically and in the geometric visualization.

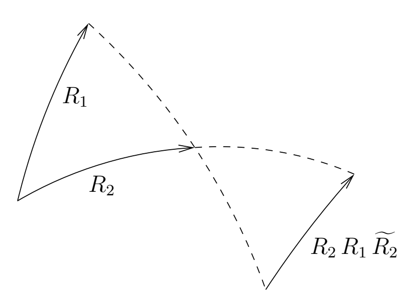
   Answer: In (7.9), if $\mathbf{I}_2 = \mathbf{I}_1 = \mathbf{I}$ then $\mathbf{I}_2\,\mathbf{I}_1 = -1$. Therefore $s_\perp = 0$, $c_\perp = 1$, and the formula would read:

   $$c'_t - \mathbf{I}_t s'_t = (c'_1 c'_2 + s'_1 s'_2) - (c'_2 s'_1 + c'_1 s'_2)\,\mathbf{I} = \cos((\phi_1+\phi_2)/2) - \sin((\phi_1+\phi_2)/2)\mathbf{I},$$

   which agrees with the 2D case. In the spherical image, everything takes place in one plane, and the spherical arc addition smoothly becomes circular arc addition.

9. Draw the rotated rotor $R_2\,R_1\,\widetilde{R}_2$ as an arc in the spherical image. Hint: what would you expect it to be, based on its geometric meaning? Warning: it is not simply the R_1 arc rotated over the R_2-arc!

   Answer: In the end, locally in the rotation sphere, if you have R_1 and R_2 as directed arc arrows starting at a common origin, the directed arc arrow for $R_2 R_1 \widetilde{R}_2$ is 'locally parallel' to that of R_1, starting at the point $2R_2 - R_1$.



11. Derive the formulas for the reflection of a dual blade $\mathbf{Y} = \mathbf{X}^*$ from the formulas for reflection of a directly represented blade $\mathbf{X}$. So, derive the last column of Table 7.1 from the column before. Make sure you take the dual of both input and output relative to the same unreflected pseudoscalar $\mathbf{I}_n$. *The table entry for row* $\mathbf{A}$ *and column* $\mathbf{Y}$ *is wrong in the first edition of the book. This exercise will help you correct it!*

    Answer: The direct formula $(-1)^{x(a+1)}\mathbf{A}\mathbf{X}\mathbf{A}^{-1}$ needs to be dualized. For the grades, $y = n - x$. This gives

$$
\begin{aligned}
(-1)^{x(a+1)}\mathbf{A}\mathbf{X}\mathbf{A}^{-1}\mathbf{I}_n^{-1} &= (-1)^{x(a+1)}(-1)^{a(n+1)}\mathbf{A}\mathbf{X}\mathbf{I}_n^{-1}\mathbf{A}^{-1} \\
&= (-1)^{(n-y)(a+1)+a(n-1)}\mathbf{A}\mathbf{Y}\mathbf{A}^{-1} \\
&= (-1)^{(y+1)(a+1)+n-1}\mathbf{A}\mathbf{Y}\mathbf{A}^{-1}.
\end{aligned}
$$

13. You can project onto a rotor, and get a geometrically meaningful result. Give the geometric interpretation of the projection $\mathsf{P}_R[\mathbf{x}] \equiv (x\rfloor R)\,R^{-1}$. (Hint: think 'chord'.) For rotors, it matters whether you put the inverse on the first or the last factor: what is $(\mathbf{x}\rfloor R^{-1})\,R$?

    Answer: Expand as $\frac{1}{2}(\mathbf{x} - R\,\mathbf{x}\,\widetilde{R})$. This is half the chord from $\mathsf{R}[\mathbf{x}]$ to $\mathbf{x}$. The other possibility involves a chord from the opposite rotation to $\mathbf{x}$.

15. **Bonus question!** Naive geometric intuition suggests that one could test whether a blade $\mathbf{X}$ is contained in a blade $\mathbf{A}$ by verifying that the reflection of $\mathbf{X}$ in $\mathbf{A}$ is identical to $\mathbf{X}$. However, this is wrong: a counterexample is provided by $\mathbf{X} = \mathbf{e}_1\,\mathbf{e}_2$ and $\mathbf{A} = \mathbf{e}_3\,\mathbf{e}_4$. Describe the geometry of this situation. The test cannot be fixed by demanding that $\mathbf{X}$ and $\mathbf{A}$ have at least one common factor, so that $\mathbf{X} \wedge \mathbf{A} = 0$. Design a counterexample for that.

    A correct test for containment can be constructed by embedding both blades in a Euclidean metric (this avoids problems with null blades). Then $\mathbf{X} \subseteq \mathbf{A}$ if and only if $\mathbf{X} \wedge (\mathbf{X}\rfloor\mathbf{A}) \neq 0$.

    Answer: $(\mathbf{e}_3\mathbf{e}_4)(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_4\mathbf{e}_3) = (\mathbf{e}_1\mathbf{e}_2)$, so the reflection is equal to the original, but the two blades are $\mathbf{X} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$ and $\mathbf{A} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_5\mathbf{e}_6$.

Chapter 8

Geometric Differentiation

8.9 Exercises

8.9.1 Drills

1. Compute the radius of the tangent circle for the circular motion $\mathbf{r}(\tau) = \exp(-\mathbf{I}\tau)\,\mathbf{e}_1$ in the plane $\mathbf{I} = \mathbf{e}_1 \wedge \mathbf{e}_2$, at the general location $\mathbf{r}(\tau)$.

   Answer: That radius is of course 1, but it is interesting to follow the computation using the procedure on page 223. We compute the various elements – you should draw them, to see how the differentiation changes the directions of the various derivatives.

$$\begin{aligned}
\mathbf{r}(\tau) &= \exp(-\mathbf{I}\tau)\,\mathbf{e}_1 \\
\dot{\mathbf{r}}(\tau) &= \exp(-\mathbf{I}\tau)\,(-\mathbf{I})\,\mathbf{e}_1 = \exp(-\mathbf{I}\tau)\,(-\mathbf{e}_1\,\mathbf{e}_2\,\mathbf{e}_1) = \exp(-\mathbf{I}\tau)\,\mathbf{e}_2 \\
\ddot{\mathbf{r}}(\tau) &= \exp(-\mathbf{I}\tau)\,(-\mathbf{I})\,\mathbf{e}_2 = \exp(-\mathbf{I}\tau)\,(-\mathbf{e}_1)
\end{aligned}$$

   Then

$$\begin{aligned}
\rho^2 &= \Big(\frac{\dot{\mathbf{r}}^3}{\dot{\mathbf{r}} \wedge \ddot{\mathbf{r}}}\Big)^2 \\
&= \Big(\frac{\exp(-\mathbf{I}\tau)\,\mathbf{e}_2\;\exp(-\mathbf{I}\tau)\,\mathbf{e}_2\;\exp(-\mathbf{I}\tau)\,\mathbf{e}_2}{(\exp(-\mathbf{I}\tau)\,\mathbf{e}_2) \wedge (\exp(-\mathbf{I}\tau)\,(-\mathbf{e}_1))}\Big)^2 \\
&= \Big(\frac{\exp(-\mathbf{I}\tau)\,\mathbf{e}_2^3}{\mathbf{e}_2 \wedge (-\mathbf{e}_1)}\Big)^2 \\
&= \big(\exp(-\mathbf{I}\tau)\,(\mathbf{e}_2\,(-\mathbf{e}_1 \wedge \mathbf{e}_2)\big)^2 = \big(\exp(-\mathbf{I}\tau)\,\mathbf{e}_1)\big)^2 \\
&= \exp(-\mathbf{I}\tau)\,\mathbf{e}_1\;\exp(-\mathbf{I}\tau)\,\mathbf{e}_1 = \mathbf{e}_1\;\exp(\mathbf{I}\tau)\;\exp(-\mathbf{I}\tau)\,\mathbf{e}_1 = \mathbf{e}_1\,\mathbf{e}_1 = 1
\end{aligned}$$

2. Compute the following derivatives.

   (a) $(\mathbf{a} * \partial_{\mathbf{x}})\,\mathbf{x}^3$

   Answer: ¿From first principles or by Table 8.1:

$$(\mathbf{a} * \partial_{\mathbf{x}})\,\mathbf{x}^3 = 2\,\mathbf{a}\,\mathbf{x}^2 + \mathbf{x}\,\mathbf{a}\,\mathbf{x} = 2\,(\mathbf{a}\cdot\mathbf{x})\,\mathbf{x} + \mathbf{a}\,\mathbf{x}^2.$$

   (b) $\partial_{\mathbf{x}}\,\mathbf{x}^3$

   Answer: The answer is $(m+2)\,\mathbf{x}^2$, which should look familiar for the 1-dimensional case $m = 1$, as essentially scalar calculus. We can derive this result per coordinate of $\partial_{\mathbf{x}}$ using the previous result:

$$\begin{aligned}\partial_{\mathbf{x}}\,\mathbf{x}^3 &= \sum_i \mathbf{e}^i\,(\mathbf{e}_i*\partial_{\mathbf{x}})\mathbf{x}^3 = \sum_i \mathbf{e}^i\,(2(\mathbf{e}_i\cdot\mathbf{x})\,\mathbf{x}+\mathbf{e}_i\,\mathbf{x}^2)\\ &= 2\sum_i \mathbf{e}^i\,(\mathbf{e}_i\cdot\mathbf{x})\,\mathbf{x}+\sum_i \mathbf{e}^i\,\mathbf{e}_i\,\mathbf{x}^2\\ &= 2\,\mathbf{x}^2+\big(\sum_i \mathbf{e}^i\cdot\mathbf{e}_i+\sum_i \mathbf{e}^i\wedge\mathbf{e}_i\big)\,\mathbf{x}^2 = 2\mathbf{x}^2+m\mathbf{x}^2+0 \quad \text{by (3.35)}.\end{aligned}$$

A better derivation is to use the product rule and Table 8.1, since that is a coordinate-free method:

$$\begin{aligned}\partial_{\mathbf{x}}\,\mathbf{x}^3 &= \dot\partial_{\mathbf{x}}\,\dot{\mathbf{x}}\,\mathbf{x}^2+\dot\partial_{\mathbf{x}}\,\mathbf{x}\,\dot{\mathbf{x}}\,\mathbf{x}+\dot\partial_{\mathbf{x}}\,\mathbf{x}^2\,\dot{\mathbf{x}}\\ &= 2\dot\partial_{\mathbf{x}}\,\dot{\mathbf{x}}\,\mathbf{x}^2+\dot\partial_{\mathbf{x}}\,(2\dot{\mathbf{x}}\cdot\mathbf{x}-\dot{\mathbf{x}}\,\mathbf{x})\,\mathbf{x}\\ &= \dot\partial_{\mathbf{x}}\,\dot{\mathbf{x}}\,\mathbf{x}^2+2\dot\partial_{\mathbf{x}}\,(\dot{\mathbf{x}}\cdot\mathbf{x})\,\mathbf{x}\\ &= (m+2)\,\mathbf{x}^2.\end{aligned}$$

(c) $(\mathbf{a}*\partial_{\mathbf{x}})\,(\mathbf{x}\,\mathbf{b}/\mathbf{x})$

Answer: ¿From first principles or by Table 8.1 and the product rule:

$$\mathbf{a}\,\mathbf{b}/\mathbf{x}+\mathbf{x}\,\mathbf{b}\,(-\mathbf{x}^{-1}\,\mathbf{a}\,\mathbf{x}^{-1}) = \mathbf{x}\,(\mathbf{x}^{-1}\,\mathbf{a}\,\mathbf{b}-\mathbf{b}\,\mathbf{x}^{-1}\,\mathbf{a})/\mathbf{x} = 2\mathbf{x}\,\big(\mathbf{b}\rfloor(\mathbf{x}^{-1}\,\mathbf{a})\big)/\mathbf{x}$$

The answer is geometrically encompassing. For instance, if $\mathbf{a}$ is parallel to $\mathbf{x}$, the derivative is zero, confirming the expectation that such longitudinal changes in $\mathbf{x}$ do not affect the result of $\mathbf{x}\,\mathbf{b}/\mathbf{x}$. Contrast the result with that of $(\mathbf{a}*\partial_{\mathbf{x}})\,(\mathbf{b}\,\mathbf{x}/\mathbf{b})$ given in Table 8.1.

(d) $\partial_{\mathbf{x}}\,(\mathbf{x}\,\mathbf{b}/\mathbf{x})$

Answer:

$$\begin{aligned}&\partial_{\mathbf{x}}\,\mathbf{x}\,\mathbf{b}\,\mathbf{x}/\|\mathbf{x}\|^2 =\\ &\quad= \dot\partial_{\mathbf{x}}\,\dot{\mathbf{x}}\,\mathbf{b}\,\mathbf{x}/\|\mathbf{x}\|^2+\dot\partial_{\mathbf{x}}\,\mathbf{x}\,\mathbf{b}\,\dot{\mathbf{x}}/\|\mathbf{x}\|^2+\dot\partial_{\mathbf{x}}\,\mathbf{x}\,\mathbf{b}\,\mathbf{x}/\|\dot{\mathbf{x}}\|^2\\ &\quad= \dot\partial_{\mathbf{x}}\,\dot{\mathbf{x}}\,\mathbf{b}\,\mathbf{x}/\|\mathbf{x}\|^2+\dot\partial_{\mathbf{x}}\,\big(2(\dot{\mathbf{x}}\cdot\mathbf{b})\,\mathbf{x}-2(\dot{\mathbf{x}}\cdot\mathbf{x})\,\mathbf{b}+\dot{\mathbf{x}}\,\mathbf{x}\,\mathbf{b}\big)/\|\mathbf{x}\|^2+\dot\partial_{\mathbf{x}}\|\dot{\mathbf{x}}\|^{-2}\,\mathbf{x}\,\mathbf{b}\,\mathbf{x}\\ &\quad= m\mathbf{b}\,\mathbf{x}^{-1}+(2\mathbf{b}\,\mathbf{x}^{-1}-2\mathbf{x}^{-1}\,\mathbf{b}+m\mathbf{x}^{-1}\,\mathbf{b})-2\mathbf{b}\,\mathbf{x}^{-1}\\ &\quad= m\mathbf{b}\,\mathbf{x}^{-1}+(m-2)\mathbf{x}^{-1}\,\mathbf{b} = 2(m-1)\mathbf{x}^{-1}\cdot\mathbf{b}-2\mathbf{x}^{-1}\wedge\mathbf{b}.\end{aligned}$$

(e) $\dot{\mathbf{x}}\,\dot\partial_{\mathbf{x}}$

Answer: $\sum_i \mathbf{e}_i x^i \sum_j \mathbf{e}^j \frac{\partial}{\partial x^j} = \sum_i \mathbf{e}_i\mathbf{e}^i = \sum_i \mathbf{e}_i\cdot\mathbf{e}^i+\sum_i \mathbf{e}_i\wedge\mathbf{e}^i = \sum_i 1+0 = m$, using (3.35).

(f) $\dot{\mathbf{x}}\wedge\dot\partial_{\mathbf{x}}$

Answer: $(\sum_i \mathbf{e}_i x^i)\wedge(\sum_j \mathbf{e}^j \frac{\partial}{\partial x^j}) = \sum_i \mathbf{e}_i\wedge\mathbf{e}^i = 0$ by (3.35).

(g) $\dot{\mathbf{x}}\cdot\dot\partial_{\mathbf{x}}$.

Answer: $(\sum_i \mathbf{e}_i x^i)\cdot(\sum_j \mathbf{e}^j \frac{\partial}{\partial x^j}) = \sum_i \mathbf{e}_i\cdot\mathbf{e}^i = \sum_i 1 = m$.

3. Show that the coordinate vectors are related to differentiation through $\mathbf{e}_k = \frac{\partial}{\partial x^k}\mathbf{x}$.

   Answer: $\frac{\partial}{\partial x^k}\mathbf{x} = \frac{\partial}{\partial x^k}\sum_i x^i\mathbf{e}_i = \delta^i_k\mathbf{e}_i = \mathbf{e}_k$.

4. Show that the reciprocal frame vectors are the gradients of coordinate functions: $\mathbf{e}^k = \partial_{\mathbf{x}}\,x^k$

   Answer: $\partial_{\mathbf{x}} x^k = \sum_i \mathbf{e}^i \frac{\partial}{\partial x^i} x^k = \sum_i \mathbf{e}^i\delta^k_i = \mathbf{e}^k$

8.9.2 Structural Exercises

1. Prove the Jacobi identity (8.2) and relate it to non-associativity of the bivector algebra.

   Answer: The proof is a straightforward matter of writing out the various terms and canceling algebraically. The non-associativity was already indicated the bottom equation on page 215, which shows to what measure that $(A \times B) \times C$ deviates from $A \times (B \times C)$.

3. The *Baker-Campbell-Hausdorff formula* writes the product of two exponentials as a third, and gives a series expansion of its value:

   $$e^C = e^A\, e^B$$

   with
   $$C = A + B + A \times B + \tfrac{1}{3}\,(A \times (A \times B) + B \times (B \times A)) + \cdots.$$

   Show that these first terms of the series are correct. This formula again shows the importance of the commutator $A \times B$ in quantifying the difference with fully commuting variables. We should warn you that the general terms of the series are more complicated than the first few suggest.

   Answer: This is a straightforward matter of writing out the exponentials in a Taylor series and collecting terms of similar grades. There is no closed-form expression for the general term, the problem is discussed in `http://en.wikipedia.org/wiki/Baker-Campbell-Hausdorff_formula`.

5. Justify the following form of Taylor's expansion formula of a function F around the location $\mathbf{x}$:
   $$F(\mathbf{x} + \mathbf{a}) = e^{\mathbf{a} * \partial_{\mathbf{x}}}\, F(\mathbf{x}),$$

   where you can interpret the exponent in a natural manner as a symbolic expansion instruction.

   Answer: This is a rather administrative exercise:

   $$\begin{aligned} e^{\mathbf{a} * \partial_{\mathbf{x}}}\, F(\mathbf{x}) &= F(\mathbf{x}) + (\mathbf{a} * \partial_{\mathbf{x}})\, F(\mathbf{x}) + \tfrac{1}{2!}(\mathbf{a} * \partial_{\mathbf{x}})(\mathbf{a} * \partial_{\mathbf{x}})\, F(\mathbf{x}) + \cdots \\ &= F(\mathbf{x}) + \sum_i a_i \tfrac{\partial}{\partial x^i} F(\mathbf{x}) + \tfrac{1}{2!} \sum_{i,j} a_i a_j \tfrac{\partial}{\partial x^i} \tfrac{\partial}{\partial x^j} F(\mathbf{x}) + \cdots \\ &= F(\mathbf{x} + \mathbf{a}) \end{aligned}$$

Chapter 11

The Homogeneous Model

11.12 Exercises

11.12.1 Drills

Compute the 2-blades corresponding to the lines gives by the data below. Which of the lines are the same, considered as weighted oriented elements of geometry, which are the same as offset subspaces?

1. Two points at locations $\mathbf{e}_1$ and $\mathbf{e}_2$.

   Answer: $(e_0 + \mathbf{e}_1) \wedge (e_0 + \mathbf{e}_2) = (e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$.

2. A point at location $\mathbf{e}_1$ and a direction $(\mathbf{e}_2 - \mathbf{e}_1)$.

   Answer: $(e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$, same as (1) in both meanings.

3. A point at location $\mathbf{e}_2$ and a direction $(\mathbf{e}_2 - \mathbf{e}_1)$.

   Answer: $(e_0 + \mathbf{e}_2) \wedge (\mathbf{e}_2 - \mathbf{e}_1) = (e_0 + \mathbf{e}_2 - (\mathbf{e}_2 - \mathbf{e}_1)) \wedge (\mathbf{e}_2 - \mathbf{e}_1) = (e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$, same as (1) in both meanings.

4. Two points with locations $2(\mathbf{e}_2 - \mathbf{e}_1)$ and $3(\mathbf{e}_2 - \mathbf{e}_1)$.

   Answer: $(e_0 + 2(\mathbf{e}_2 - \mathbf{e}_1)) \wedge (e_0 \wedge 3(\mathbf{e}_2 - \mathbf{e}_1)) = e_0 \wedge (\mathbf{e}_2 - \mathbf{e}_1)$, different from (1) in both meanings.

5. A point at location $\mathbf{e}_1$ and a direction $2(\mathbf{e}_2 - \mathbf{e}_1)$.

   Answer: $(e_0 + \mathbf{e}_1) \wedge (2(\mathbf{e}_2 - \mathbf{e}_1)) = 2(e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$ same subspace as (1), but a different weight.

6. A unit point at location $\mathbf{e}_1$ and a point with weight 2 at location $\mathbf{e}_2$.

   Answer: $(e_0 + \mathbf{e}_1) \wedge 2(e_0 + \mathbf{e}_2) = 2(e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$ same subspace as (1), but a different weight; same as (5), in both meanings.

11.12.2 Structural Exercises

1. Let an orthonormal coordinate system $\{\mathbf{e}_i\}_{i=1}^3$ be given in 3-dimensional Euclidean space. Compute the support vector of the line with direction $\mathbf{u} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, through the point $\mathbf{p} = \mathbf{e}_1 - 3\mathbf{e}_2$. What is the distance of the line to the origin?

   Answer: $\mathbf{d} = (\mathbf{p} \wedge \mathbf{u}/)\mathbf{u} = (11\mathbf{e}_1 - 8\mathbf{e}_2 - 5\mathbf{e}_3)/6$, and $\sqrt{35/6}$.

3. Show that the support vector $\mathbf{d}$ of a k-flat is the rejection of the position vector of an arbitrary point p on it by the k-direction $\mathbf{A}$.

   Answer: The support vector is defined through $\mathbf{p} \wedge \mathbf{A} = \mathbf{d}\,\mathbf{A}$, so that $\mathbf{d} = (\mathbf{p} \wedge \mathbf{A})/\mathbf{A}$. That is indeed the rejection of $\mathbf{p}$ by $\mathbf{A}$, the component of $\mathbf{p}$ perpendicular to $\mathbf{A}$.

5. Three points a, b, c form a plane, and these points can be used to address any other point x in that plane as a linear combination:

$$x = \alpha\, a + \beta\, b + \gamma c.$$

   Using normalized points, one can do this with an affine combination. The resulting scalars α, β, γ are called *barycentric coordinates* (literally, 'weight-based'). Compute α, β, γ in terms of the points a, b, c, and express the result using the relative vectors $\mathbf{a} = a - c$, $\mathbf{b} = b - c$ and $\mathbf{x} = x - c$. This should give you:

$$\alpha = \frac{\mathbf{x} \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}}, \quad \beta = \frac{\mathbf{x} \wedge \mathbf{a}}{\mathbf{b} \wedge \mathbf{a}}, \quad \gamma = 1 - \frac{\mathbf{x} \wedge (\mathbf{b} - \mathbf{a})}{\mathbf{a} \wedge \mathbf{b}}. \tag{11.18}$$

   Interpret the result geometrically in terms of areas in the plane (most easily seen when x is inside the triangle formed by a, b and c). What are the barycentric coordinates of the center of gravity?

   These barycentric coordinates can be used to interpolate any scalar property ϕ given at each of the vertices of a triangle to an intermediate ϕ_x value at x, through:

$$\phi_x = \alpha\, \phi_a + \beta\, \phi_b + \gamma\, \phi_c \tag{11.19}$$

   This equation will be used in the ray tracer of Chapter 23.

   Answer: $\alpha = \frac{x \wedge b \wedge c}{a \wedge b \wedge c} = \frac{\mathbf{x} \wedge \mathbf{b} \wedge c}{\mathbf{a} \wedge \mathbf{b} \wedge c} = \frac{(\mathbf{x} \wedge \mathbf{b})\, e_0}{(\mathbf{a} \wedge \mathbf{b})\, e_0} = \frac{\mathbf{x} \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}}$. This is the area of the triangle to x 'across from a' divided by the area of the total triangle. The center of gravity is at $(1, 1, 1)/3$.

   Bonus question: Show that the support vector of a plane through three points p, q, r equals (11.4):

$$\mathbf{d} = \frac{\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}}{\mathbf{p} \wedge \mathbf{q} + \mathbf{q} \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{p}}.$$

   Note that this expression is a fully computable in the Euclidean space, yet its (rather straightforward) derivation belongs more properly to the homogeneous model.

   Answer: To phrase the problem in terms of homogeneous coordinates, we use the definition $p \wedge q \wedge r = d\big((q - p) \wedge (r - p)\big)$. The left hand side is $p \wedge (q - p) \wedge (r - p) = p \wedge (\mathbf{q} - \mathbf{p}) \wedge (\mathbf{r} - \mathbf{p})$. The attitude is more symmetrical and Euclidean than it appears: $(q - p) \wedge (r - p) = (\mathbf{q} - \mathbf{p}) \wedge (\mathbf{r} - \mathbf{p}) = \mathbf{q} \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{p} + \mathbf{p} \wedge \mathbf{q}$. Now taken the non-$e_0$-part of both sides of the defining equation produces the expression for $\mathbf{d}$.

7. In the parametric equation for an offset flat (11.5), the vector $\mathbf{x}$ determines the values of the λ_i uniquely. Compute a formula for λ_i. (Hint: eliminate the other λ_j, with $j \neq i$, by suitably chosen outer products with $\mathbf{a}_j$ vectors. Alternatively, use the idea of a reciprocal basis from Section 3.8.)

   Answer: For λ_1, we take the outer product with $\mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k$. This gives terms of both sides proportional to the pseudoscalar, so division is well-defined, resulting in:

$$\lambda_1 = \frac{(\mathbf{x} - \mathbf{p}) \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k}{\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k}.$$

The rest is similar.

9. Construct the dual representation of the midplane between two points p and q.

   Answer: The plane goes through the point at $\frac{1}{2}(\mathbf{p}+\mathbf{q})$ and has attitude $\pm(\mathbf{q}-\mathbf{p})^\star$. So the direct representation is $\pm\frac{1}{2}(p+q)\wedge(q-p)^\star$ (note the Euclidean dual!) and the dual representation is $\pm\big(\frac{1}{2}(p+q)\wedge(q-p)^\star\big)^* \propto \frac{1}{2}(p+q)\rfloor\big((q-p)\,e_0^{-1}\big)$

11. The meet of two skew lines $p\wedge\mathbf{u}$ and $q\wedge\mathbf{v}$ can be computed as $M^*\rfloor L$. Verify the steps in the following derivation of (11.9) using this formula.

$$
\begin{aligned}
(p\wedge\mathbf{u})\cap(q\wedge\mathbf{v}) &= \big((q\wedge\mathbf{v})\rfloor(\mathbf{I}_3^{-1}\wedge e_0{}^{-1})\big)\rfloor(p\wedge\mathbf{u})\\
&= \big(q\rfloor(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\wedge e_0{}^{-1}\big)\rfloor(p\wedge\mathbf{u})\\
&= \big(\mathbf{q}\rfloor(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\wedge e_0{}^{-1}+(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\big)\rfloor(p\wedge\mathbf{u})\\
&= \big(\mathbf{q}\rfloor(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\big)\rfloor\mathbf{u}+(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\rfloor(\mathbf{p}\wedge\mathbf{u})\\
&= \mathbf{u}\rfloor\big(\mathbf{q}\rfloor(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\big)+(\mathbf{p}\wedge\mathbf{u})\rfloor(\mathbf{v}\rfloor\mathbf{I}_3^{-1})\\
&= (\mathbf{u}\wedge\mathbf{q}\wedge\mathbf{v})\rfloor\mathbf{I}_3^{-1}+(\mathbf{p}\wedge\mathbf{u}\wedge\mathbf{v})\rfloor\mathbf{I}_3^{-1}\\
&= ((\mathbf{p}-\mathbf{q})\wedge\mathbf{u}\wedge\mathbf{v})\rfloor\mathbf{I}_3^{-1}
\end{aligned}
$$

13. In Section 11.7.2, we stated: 'in a plane with anti-clockwise orientation, the positive side of the line is on your left when you look along its direction'. Convince yourself that this statement gives the same positive side independent of whether you look at the plane from above or below, so that it is a truly geometrically invariant definition. That is good, for it would be useless otherwise.

    Answer: Only you can convince yourself.

15. Show explicitly that the determinants of the translation formulas (11.13) and (11.14) for a flat and for a flat dual both equal 1.

    Answer: The determinant is the scaling of the transformation of the pseudoscalar, so we need to transform that. This is like the previous exercise. Using (11.13) we get: $\mathsf{T}_\mathbf{t}[e_0\,\mathbf{I}_n]=e_0\,\mathbf{I}_n+\mathbf{t}\wedge(e_0^{-1}\rfloor(e_0\,\mathbf{I}_n))=e_0\,\mathbf{I}_n+\mathbf{t}\wedge\mathbf{I}_n)=e_0\,\mathbf{I}_n$. The dual is $(e_0\mathbf{I}_n)^*=1$, and using (11.14) we get $\mathsf{T}^*_\mathbf{t}[1]=1-e_0^{-1}\wedge(\mathbf{t}\rfloor1)=1$. Alternatively, the dual result is immediate when we realize that any linear transformation preserves a scalar.

17. There is a way to patch up the homogeneous model so that translation becomes representable in a versor-like form, and you may find this used in the somewhat older literature (such as [4]). It uses a different metric in which $e_0\cdot e_0=0$, and represents a point at location $\mathbf{x}$ as $1+\mathbf{x}I_4$, where $I_4=e_0\mathbf{I}_3$ is the pseudoscalar of the homogeneous representation space. Show that in this approach, the element $(1+\mathbf{t}\,I_4/2)$ acts as a translation versor on points. (**Erratum: this is not correct, see bottom of exercise.**) Therefore the translation of points, at least, can be represented as a versor — and with it, general rigid body motions on points.

    However, the representation of the higher grade objects (such as lines and planes) in such a model is *ad hoc*, in that various objects are not related to each other by an outer product-like spanning operation, or a meet-like product for intersection. As a consequence, the versor form of the translation is not perfect: some objects should be translated as $T\,X\,\widetilde{T}$, others as $T\,X\,T$, whereas one would have hoped that such fundamental operations would be independent of their argument.

Since the invention of the conformal model of Chapter 13 (which fixes all these defects by using null vectors in a different manner), this '*motor algebra*' has fallen into disuse, and we mention it here only for completeness.

Erratum: *This exercise is flawed. Even for the translation of a point x, one needs to use a non-versor-like sandwiching $T\,x\,T$. Show that.*

Answer: Note that the erratum illuminates the statement at the end: scalars transform differently from points.
The translation element $T = 1 + \mathbf{t}\,I_4/2$ can sandwich a point $x = 1 + \mathbf{x}\,I_4$ to produce the translated point $1 = (\mathbf{x} + \mathbf{t})\,I_4$, but we have to be careful about the signs:

$$1 + (\mathbf{x} + \mathbf{t})\,I_4 = (1 + \mathbf{t}\,I_4/2)\,(1 + \mathbf{x}\,I_4)\,(1 + \mathbf{t}\,I_4/2).$$

No higher terms appear because of the special metric in which $I_4^2 = 0$. Therefore a translation is made as $T\,x\,T$. This is not actually a versor product; that would involve inverse or reverse of T, which are: $T^{-1} = \widetilde{T} = 1 - \mathbf{t}\,I_4$. Yet in the conformal model, a similar trick will work: again the metric makes sure no higher order terms appear, and moreover the reverse versor is used.

19. You are to draw a sequence of equidistant telegraph poles along a straight road in a picture showing the landscape seen in a bird's eye view, with the horizon 6 cm from the first pole, and the separation between first and second pole 1 cm (see Figure 11.13. Compute where the third pole should be. Extend this to computing the location of the k-th pole. (Hint: Compute the cross ratio of the first two poles to the point at infinity in a 'straight' photograph. Then realize that the cross ratio is a projective invariant.)

    Answer: Let us solve this for pole k, numbering the poles from zero. We compute a cross ratio as in Figure 11.8 for p as pole 0, q as pole 1, r as pole k and s as the pole at infinity. That cross ratio, as a function of k, is $\big((1)(\infty - k)\big)/\big((k-1)(\infty)\big) = 1/(k-1)$. (If you are uncomfortable with $\infty/\infty = 1$, just use the pole numbered googolplex instead.) Since the cross-ratio is a projective invariant, it should be the same in the picture we draw. If pole k is drawn as a distance x from pole 0 in the picture, we should therefore have: $\big((1)(6 - x)\big)/\big((x-1)(6)\big) = 1/(k-1)$. It follows that $x = 6k/(k+5)$ is the location, in centimeters, of pole k. The third pole ($k = 2$) should therefore be drawn $5/7$ cm further along the line.

21. Redo some of the orthogonal projection examples in the vector space model of a 3-dimensional Euclidean space using the meet interpretation of (11.17).

    Answer: As an example, we can take a line through the origin characterized by a vector $\mathbf{x}$, and a plane through the origin characterized by a unit bivector $\mathbf{A}$, in 3D. Then the usual projection formula is $(\mathbf{x}\rfloor\mathbf{A})/\mathbf{A} = -(\mathbf{x}\rfloor\mathbf{A})\rfloor\mathbf{A}$. According to (11.17), this should be minus the meet of the plane $\mathbf{A}$ with $\mathbf{x} \wedge \mathbf{A}^{-*} = -\mathbf{x} \wedge \mathbf{a}$, with $\mathbf{a} \equiv \mathbf{A}^*$ the usual normal vector of the plane. So the result is equal to the meet of the planes $\mathbf{A}$ and $\mathbf{x} \wedge \mathbf{a}$. That intersection certainly produces a line with the correct carrier. Do a simple example $\mathbf{A} = \mathbf{e}_1 \wedge \mathbf{e}_2$ (so that $\mathbf{a} = \mathbf{e}_3$) and $\mathbf{x} = \mathbf{e}_1$ to see that the magnitude and orientation are correct as well (if you use the unit pseudoscalar $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ as join).

Chapter 12

Applications of the Homogeneous Model

12.5 Exercises

12.5.1 Structural Exercises

1. Table 12.1 contains the case in which a line $\{\mathbf{a}, \mathbf{m}\}$ is extended to a plane by an additional direction $\mathbf{n}$, to form the plane $[\mathbf{a} \times \mathbf{n} : \mathbf{n} \cdot \mathbf{m}]$. Demonstrate the correctness of this formula, by representing the spanning $L \wedge \mathbf{n}$ in terms of the Plücker coordinates.

   Erratum: *Due to a somewhat unfortunate convention in [57], equation (12.3) should have an overall minus sign, and the entry* $\{\mathbf{a}, \mathbf{m}\}$ *in Table 12.1 represents the line with direction* $-\mathbf{a}$ *and moment* $-\mathbf{m}$.

   Answer: According to (12.4), the line with direction $-\mathbf{a}$ and moment $-\mathbf{m}$ is represented by $L = -\mathbf{a}\, e_0 - \mathbf{m}\, \mathbf{I}_3 = -\mathbf{a} \wedge e_0 - \mathbf{m} \rfloor \mathbf{I}_3$. The plane then is

$$
\begin{aligned}
L \wedge \mathbf{n} &= -\mathbf{a} \wedge e_0 \wedge \mathbf{n} - (\mathbf{m} \rfloor \mathbf{I}_3) \wedge \mathbf{n} \\
&= (\mathbf{a} \wedge \mathbf{n})\, e_0 - \mathbf{n} \wedge (\mathbf{m} \rfloor \mathbf{I}_3) \\
&= \big((\mathbf{a} \wedge \mathbf{n}) \mathbf{I}_3^{-1}\big)\, \mathbf{I}_3\, e_0 - (\mathbf{n} \cdot \mathbf{m}) \rfloor \mathbf{I}_3 \\
&= -(\mathbf{a} \times \mathbf{n})\, e_0\, \mathbf{I}_3 - (\mathbf{n} \cdot \mathbf{m})\, \mathbf{I}_3.
\end{aligned}
$$

   Comparison to the direct plane equation on pg. 331 shows that the normal vector is $\mathbf{a} \times \mathbf{n}$, and the origin distance $-\mathbf{n} \cdot \mathbf{m}$. The plane is therefore represented in Plücker coordinates as $[\mathbf{a} \times \mathbf{n}, \mathbf{n} \cdot \mathbf{m}]$, in agreement with the entry in Table 12.1.

3. Knowing some of the standard formulas in geometric algebra, you may recognize that the central projection formula (12.9) is not unlike the usual orthogonal projection formula onto a line with direction $\mathbf{a}$, which maps $\mathbf{x}$ to $(\mathbf{x} \cdot \mathbf{a})\, \mathbf{a}^{-1}$. Demonstrate that we can consider the central projection $\mathbf{x}' \mapsto \mathbf{x}/(\mathbf{f}^{-1} \cdot \mathbf{x})$ as the fixed vector $\mathbf{f}$ gets inverted, projected onto the variable vector $\mathbf{x}$, and then re-inverted, to produce $\mathbf{x}'$. This interpretation of the formula generalizes to substituting $\mathbf{x}$ by a line or plane (just replace the inner product by the contraction): it then produces the support vector of the projected line or plane.

   (Hint: show that the $\mathbf{x}'$ satisfies: $\mathbf{x}'^{-1} = (\mathbf{f}^{-1} \cdot \mathbf{x})\, \mathbf{x}^{-1} = (\mathbf{f}^{-1} \cdot \mathbf{x}^{-1})\, \mathbf{x}$. and interpret.)

   Answer: The steps described are $\mathbf{f} \to \mathbf{f}^{-1} \to (\mathbf{f}^{-1} \cdot \mathbf{x})\, \mathbf{x}^{-1} \to \mathbf{x}/(\mathbf{f}^{-1} \cdot \mathbf{x})$, so indeed provide the result.

5. Equation (12.9) gives the projection of a unit 3D point at location $\mathbf{x}$ to become a point at location $\mathbf{x}'$, which is on the image plane, but it is not expressed yet in image plane coordinates $\underline{\mathbf{x}}$. Show that the mapping from the 3D point at $\mathbf{x}$ to the image point at $\underline{\mathbf{x}} = \mathbf{x}' - \mathbf{f}$ can be written as

$$\mathbf{x} \quad \mapsto \quad \frac{\mathbf{f}\,(\mathbf{f}^{-1} \wedge \mathbf{x})}{\mathbf{f}^{-1} \cdot \mathbf{x}}$$

Interpret this expression geometrically, especially the numerator.

Answer: This is straightforward simplification:

$$\underline{\mathbf{x}} = \frac{\mathbf{x}}{\mathbf{f}^{-1} \cdot \mathbf{x}} - \mathbf{f} = \frac{\mathbf{f}\,\mathbf{f}^{-1}\,\mathbf{x} - \mathbf{f}\,(\mathbf{f}^{-1} \cdot \mathbf{x})}{\mathbf{f}^{-1} \cdot \mathbf{x}} = \frac{\mathbf{f}\,(\mathbf{f}^{-1} \wedge \mathbf{x})}{\mathbf{f}^{-1} \cdot \mathbf{x}}.$$

Geometrically, this is the normalized rejection of the variable vector $\mathbf{x}$ by the fixed vector $\mathbf{f}^{-1}$. That rejection (before normalization) lies on the eyeball sphere of the previous exercise.

Chapter 13

The Conformal Model: Operational Euclidean Geometry

13.9 Exercises

13.9.1 Drills

These drills intend to familiarize you with the form of common geometric elements and their parameters in the conformal model. We recommend doing them by hand first, and check then with interactive software later.

1. Give the representation of a point p_1 with weight 2 at location $\mathbf{e}_1 + \mathbf{e}_2$.

   Answer: $2(o + (\mathbf{e}_1 + \mathbf{e}_2) + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)^2 \infty) = 2(o + (\mathbf{e}_1 + \mathbf{e}_2) + \infty)$.

2. Give the representation of a point p_2 with weight -1 at location $\mathbf{e}_1 + \mathbf{e}_3$, and compute its distance to p_1.

   Answer: $-(o + (\mathbf{e}_1 + \mathbf{e}_3) + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3)^2 \infty) = -(o + (\mathbf{e}_1 + \mathbf{e}_3) + \infty)$. The squared distance d to p_1 follows from

$$\begin{aligned} d &= -2\,\frac{p_1 \cdot p_2}{(-\infty \cdot p_1)(-\infty \cdot p_2)} \\ &= -2\,\frac{\big(2(o + (\mathbf{e}_1 + \mathbf{e}_2) + \infty\big) \cdot \big(-(o + (\mathbf{e}_1 + \mathbf{e}_3) + \infty\big)}{(2)(-1)} \\ &= -2\,\big(2\,o \cdot \infty + (\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{e}_1 + \mathbf{e}_3)\big) = -2(-2+1) = 2. \end{aligned}$$

3. Give the representation of the line L through p_1 and p_2.

   Answer: $p_1 \wedge p_2 \wedge \infty = -2(o + (\mathbf{e}_1 + \mathbf{e}_2)) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$.

4. Compute weight and direction of the line L.

   Answer: We have given it in factorized form, in which the Euclidean directional element is read as $-2(\mathbf{e}_3 - \mathbf{e}_2)$; so the weight is -2, and the conformal direction element is $-2(\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$.

5. Compute the support point on the line L.

   Answer: [[[**they only really learn to do that via Table 14.1 next chapter?**]]]

6. Give the direct representation of the plane Π through L and the unit point at the origin.

   Answer: $\Pi = L \wedge o = 2o \wedge (o + (\mathbf{e}_1 + \mathbf{e}_2)) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty = 2o \wedge (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$.

7. Compute the direction and support of the plane Π.

   Answer: The direction is clearly $2(\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$. [[[**support chapter 14?**]]]

8. Give the representation of the translation over $-\mathbf{e}_1$ of the plane Π.

   Answer: This translation only affects the location o, so that is simply $(o - \mathbf{e}_1) \wedge (\text{direction}) = 2(o - \mathbf{e}_1) \wedge (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$.

9. Compute the dual π of the plane Π. Compute its dual direction, and its moment.

   Answer: Dualization of $2o \wedge (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty$ gives

$$\begin{aligned}
&\big(2o \wedge (\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty\big)\, \mathbf{I}_n^{-1} (o \wedge \infty) \\
&= 2\Big((\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2)\Big)\, \mathbf{I}_n^{-1}\, (o \wedge \infty)\, (o \wedge \infty) \\
&= 2\Big((\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2)\Big)^{\star}.
\end{aligned}$$

10. Compute the dual of the line L.

    Answer:

$$\begin{aligned}
L^* &= (-2(o + (\mathbf{e}_1 + \mathbf{e}_2)) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty)^* \\
&= (-2(o + (\mathbf{e}_1 + \mathbf{e}_2)) \wedge (\mathbf{e}_3 - \mathbf{e}_2) \wedge \infty) \rfloor (o \wedge \mathbf{I}_n^{-1} \wedge \infty) \\
&= 2\mathbf{e}_1 \wedge \mathbf{e}_2 + 2\mathbf{e}_1 \wedge \mathbf{e}_3 + 2\mathbf{e}_1 \wedge \infty - 2\mathbf{e}_2 \wedge \infty - 2\mathbf{e}_3 \wedge \infty
\end{aligned}$$

    Comparing this answer to the dual line in the homogeneous model (12.6), you recognize the 2-blade that is dual to the direction vector, and the classical moment vector $-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$.

11. **Erratum:** *This is a nonsensical question, which has been cancelled.* Compute the distance between L and its dual. (You will have to invent a conformal formula for this yourself!)

13.9.2 Structural Exercises

1. Show that on the $\{e, \bar{e}\}$-basis, the point p of (13.3) is represented as:

$$p = \mathbf{p} + \tfrac{1}{2}\,(1 - \mathbf{p}^2)\, e + \tfrac{1}{2}\,(1 + \mathbf{p}^2)\, \bar{e}.$$

   In [33] and [15], you find the close relationship of this formula with stereographic projection spelled out, as another way of visualization the conformal model. Unfortunately, it needs the two extra dimensions, so that you can only visualize the model for a 1-dimensional Euclidean space.

   Answer: Simply substitute (13.6) and rearrange.

3. In structural exercise **??** of Section **??**, we introduced barycentric coordinates using the homogeneous model. Using the correspondence between homogeneous model and conformal model, give expressions for the barycentric coordinates in terms of conformal points.

   Answer: Just use flat elements: $\alpha = \frac{x \wedge b \wedge c \wedge \infty}{a \wedge b \wedge c \wedge \infty}$ et cetera.

5. For a pure translation versor T, the logarithm is easy to determine. Show that
$$\log(T) = \tfrac{1}{2}(T - \widetilde{T}),$$
and adapt the algorithm of Figure 13.5.

Answer: T is the exponent of $-\mathbf{t}\infty/2$, which also occurs in its expansion $T = 1 - \mathbf{t}\infty/2$. When you observe that $\widetilde{T} = 1 + \mathbf{t}\infty/2$, the result is immediate.

7. Show that the ratio of two flat points $p \wedge \infty$ and $q \wedge \infty$ is a translation rotor. What is the corresponding translation vector?

Answer:

$$\begin{aligned}
(q \wedge \infty)/(p \wedge \infty) &= (q \wedge \infty)\,(p \wedge \infty) \\
&= (q\,\infty + 1)\,(p\,\infty + 1) \\
&= 1 + q\,\infty + p\,\infty + q\,\infty\,p\,\infty \\
&= 1 + q\,\infty + p\,\infty - 2q\,\infty \quad \text{(see exercise 2)} \\
&= 1 - (q - p)\infty \\
&= 1 - (\mathbf{q} - \mathbf{p})\infty,
\end{aligned}$$

Comparison with the rotor definition of Section 13.2.2 shows that the translation vector is $2(\mathbf{q} - \mathbf{p})$, i.e., twice the separation of the points.

9. Show that the ratio of two lines $p \wedge \mathbf{n} \wedge \infty$ and $q \wedge \mathbf{m} \wedge \infty$ is a general rigid body motion. What are the screw parameters?

Answer:

$$\begin{aligned}
&(q \wedge \mathbf{m} \wedge \infty)/(p \wedge \mathbf{n} \wedge \infty) = \\
&\quad= (q \wedge \mathbf{m} \wedge \infty)\,(\infty \wedge p \wedge \mathbf{n}) \\
&\quad= ((q \wedge \mathbf{m})\,\infty - \mathbf{m})\,(\infty\,(p \wedge \mathbf{n}) + \mathbf{n}) \\
&\quad= -\mathbf{mn} + (q \wedge \mathbf{m})\,\infty\,\mathbf{n} - \mathbf{m}\,\infty\,(p \wedge \mathbf{n}) \\
&\quad= \mathbf{mn} + (q \wedge \mathbf{m} \wedge \infty)\,\mathbf{n} - \mathbf{m}\,(\infty \wedge p \wedge \mathbf{n}) \\
&\quad= \mathbf{mn} + (-(q \wedge \mathbf{m})\lfloor \mathbf{n} - \mathbf{m} \rfloor (p \wedge \mathbf{n})) \wedge \infty + q \wedge \mathbf{m} \wedge \infty \wedge \mathbf{n} - \mathbf{m} \wedge \infty \wedge p \wedge \mathbf{n} \\
&\quad= \mathbf{m}\,\mathbf{n} + (\mathbf{n} \rfloor (\mathbf{q} \wedge \mathbf{m}) - \mathbf{m} \rfloor (\mathbf{p} \wedge \mathbf{n})) \wedge \infty - (\mathbf{p} - \mathbf{q}) \wedge \mathbf{m} \wedge \mathbf{n} \wedge \infty.
\end{aligned}$$

Chapter 14

New Primitives for Euclidean Geometry

14.9 Exercises

14.9.1 Drills

These drills intend to familiarize you with the form of common geometric elements and their parameters in the conformal model. We recommend doing them by hand first, and check then with interactive software later.

1. Give the direct representation of the point pair (0-sphere) P spanned by the points p_1 and p_2 at location $\mathbf{e}_1$ and $\mathbf{e}_2$, with weights 2 and -1.

   Answer:

$$2(o + \mathbf{e}_1 + \tfrac{1}{2}\infty) \wedge (-1)(o + \mathbf{e}_2 + \tfrac{1}{2}\infty) \quad = \quad -2(o + \mathbf{e}_1 + \infty/2) \wedge (\mathbf{e}_2 - \mathbf{e}_1)$$

2. Compute center and radius of P.

   Answer: We can compute the radius squared using Table 14.1 *(beware of the errata correcting signs in this table!)* as $P\widehat{P}/(\infty\rfloor P)^2 = 1/2$, and the location through $P/(-\infty\rfloor P)$ as being at $(\mathbf{e}_1 + \mathbf{e}_2)/2$.

3. Give the dual representation of P, and use it to compute radius and center.

   Answer: Let us treat this as a planar problem, with pseudoscalar $o \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \infty$.

$$\begin{aligned}
\Big(-2(o + \mathbf{e}_1 + \infty/2) \wedge (\mathbf{e}_2 - \mathbf{e}_1)\Big)^* &= -2(o + \mathbf{e}_1 + \infty/2)\rfloor\big((\mathbf{e}_2 - \mathbf{e}_1)^\star(o \wedge \infty)\big) \\
&= -2(o + \mathbf{e}_1 + \infty/2)\rfloor\big((\mathbf{e}_1 + \mathbf{e}_2)(o \wedge \infty)\big) \\
&= -2\big((o - \infty/2)\,(\mathbf{e}_1 + \mathbf{e}_2) - o \wedge \infty\big)
\end{aligned}$$

   Now apply the dual formulas from Table 14.1 to get the same results for radius and location.

4. Retrieve the locations of the original points from P (see (14.13) below).

   Answer: Point pairs are the only rounds for which one can retrieve the points that constituted them. The use of (14.13) is straightforward, and you know what the answer should be since you made P yourself.

5. Compute the carrier line of P, both in direct and dual form.
   Erratum: *carriers are only properly defined in the next chapter, Section 15.2.2.*

   Answer: The carrier of the direct round P is $\infty \wedge P = -2(o + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge \infty$. The carrier of the dual round $p = P^*$ is the dual of this, and equal to $2(\mathbf{e}_1 + \mathbf{e}_2 + \infty)$.

6. Give the direct representation of the circle K through p_1, p_2 and the unit point at location $\mathbf{e}_3$.

   Answer: Computation of all the terms gives: $-2(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge o - \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge o + \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge o + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \infty/2 - \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \infty/2 + \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \infty/2)$.

7. Compute the squared radius and the center of the circle K.

   Answer: Use Table 14.1, the result should be $2/3$ and the point at $(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/3$. The weights do not affect the geometry of the circle location, merely its own weight.

8. Give the direct representation of the sphere Σ through K and the origin.

   Answer: Computing all terms gives: $\Sigma = -2(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge o - \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge o \wedge \infty/2 + \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge o \wedge \infty/2 - \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge o \wedge \infty/2)$. The explanation is clear from the dual (next subquestion).

9. Compute the dual of Σ and read off its center and squared radius, directly from that dual representation.

   Answer: $\Sigma^* = 2(o + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/2)$, and it is clear that the center is at $(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/2$, and the radius squared is the same as that of the center point location, i.e. $3/4$ (so that it cancels the potential ∞ term precisely).

14.9.2 Structural Exercises

1. The normalized sphere through four points p, q, r, s is: $\Sigma = (p \wedge q \wedge r \wedge s)/(p \wedge q \wedge r \wedge s \wedge \infty)^*$. Show that the Euclidean vector pointing to the center of this sphere is

$$\mathbf{c} = (o \wedge \infty) \rfloor \big(o \wedge \infty \wedge \Sigma^*\big) = \big((o \wedge \infty) \rfloor \Sigma\big)^\star,$$

   Note that the final rewriting involves the Euclidean dual. The first form is the rejection of the non-Euclidean parts from the dual. It is easily implemented as simply listing the Euclidean part of the normalized sphere.

   Answer: From Section 14.1.2, we know that the relationship with the dual is $\Sigma^* = c - \frac{1}{2}\rho^2 \infty = o + \mathbf{c} + \frac{1}{2}(\mathbf{c}^2 - \rho^2)\infty$. The rejection of $o \wedge \infty$ then indeed gives $\mathbf{c}$.

3. The weight of a dual sphere σ is the weight of its center, and equal to $\infty \cdot \sigma$. Dualize this expression to discover when a sphere through the points p, q, r, s becomes zero.

   Answer: $(\infty \cdot \sigma)^* = \infty \wedge \sigma^* = \infty \wedge p \wedge q \wedge r \wedge s$. This is zero when s is in the plane $p \wedge q \wedge r \wedge \infty$, i.e., when the points are coplanar. So the degenerate sphere through four coplanar points, though a sensible algebraic object, has weight zero.

5. For a flat point $P = p \wedge \infty$, (14.13) does not work, since it then requires division by a null vector. In that case, the simplest method is to retrieve the Euclidean position vector $\mathbf{p}$ and use that to make the point p. In an implementation, the coordinates of $\mathbf{p}$ are found as the coefficients of the basis

blades $\mathbf{e}_1 \wedge \infty$, $\mathbf{e}_2 \wedge \infty$, and $\mathbf{e}_3 \wedge \infty$, divided by the coordinate of $o \wedge \infty$. Algebraically, show that

$$\mathbf{p} = -\frac{(o \wedge \infty) \rfloor (o \wedge P)}{(o \wedge \infty) \rfloor P}.$$

Answer: Just substitute $P = p \wedge \infty$:

$$-\frac{(o \wedge \infty) \rfloor (o \wedge p \wedge \infty)}{(o \wedge \infty) \rfloor (p \wedge \infty)} = \frac{(o \wedge \infty) \rfloor (o \wedge \infty \wedge \mathbf{p})}{(o \wedge \infty) \rfloor (p \wedge \infty)} = \frac{\mathbf{p} + 0 + 0}{1 + 0} = \mathbf{p}$$

7. Compute the meet of the dual circles $\kappa_1 = \mathsf{T}_{\mathbf{e}_2}[(o - \frac{1}{2}\infty)\,(-\mathbf{e}_3)]$ and $\kappa_2 = \mathsf{T}_{-\mathbf{e}_2}[(o - \frac{1}{2}\infty)\,(-\mathbf{e}_3)]$, both residing in the $\mathbf{e}_1 \wedge \mathbf{e}_2$-plane. It is a tangent vector — what is its weight, and how is that related to the geometry of the situation?

   Answer: $\kappa_1 = (o + \mathbf{e}_2) \wedge (-\mathbf{e}_3)$, $\kappa_2 = (o - \mathbf{e}_2) \wedge (-\mathbf{e}_3)$, Their meet is then $(\kappa_2 \wedge \kappa_1)^{-*} = (2o \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)^{-*} = 2\,o\,\mathbf{e}_1$. So the weight is 2, and following the computation you see that it is twice the radius of these same-size unit-weight circles. You may want to see what happens with touching circles of different size.

9. In Figure (14.7), the green line segments are part of the Voronoi diagram. The points of these segments should represent Euclidean circles. Draw these circles in the Euclidean space. Similarly, the edges of the Delaunay triangulation represent circles, but they are imaginary. Draw some of those. (For a hint, see Figure 15.8.)

   Answer: The green lines in the figure are the dual of lines connecting points, so they are of the form $(p \wedge q)^*$. If we demand to know what dual circle σ is on such a line, we have to solve $\sigma \wedge (p \wedge q)^* = 0$. By duality that is the same as $\sigma \rfloor (p \wedge q) = 0$. With p and q in general position, the resulting equation $(\sigma \cdot p)\, q - q\, (\sigma \cdot q) = 0$ leads to $\sigma \cdot p = 0$ and $\sigma \wedge q = 0$. Therefore we interpret σ as the dual representation of a circle that contains both p and q. In the Euclidean plane, such circles are easily drawn: any point of the perpendicular bisector between the points p and q is the center of such a circle.
   By contrast, the blue lines are much harder to interpret in the Euclidean plane. Let us do a simplified computation to study them. Elements σ of the blue line between p and q satisfy $\sigma \wedge p \wedge q = 0$. Therefore σ is a linear combination of p and q, and if we normalize σ this can be written as the affine combination $\sigma_\lambda = \lambda p + (1 - \lambda) q$. Since everything is translation-invariant, let us for convenience take $q = o$ to study these elements. Then we find $\sigma_\lambda = \left(o + \lambda \mathbf{p} + \frac{1}{2}\lambda^2 \mathbf{p}^2 \infty\right) + \frac{1}{2}\lambda(1 - \lambda)^2 \mathbf{p}^2 \infty$. This is an dual imaginary circle with center at $\lambda\mathbf{p}$, and the squared radius $-\lambda(1 - \lambda)\mathbf{p}^2$. These correspond precisely to the dashed purple circles in Figure 15.8. Geometrically, they are not as intuitive as the points on the green lines.

11. Extending Figure 14.8, draw pictures displaying the inner product of two spheres when the center of one is contained inside the other sphere, and when one sphere is fully contained inside the other sphere.

    Answer: These two constructons are essentially identical, we provide the second. Let sphere 2 be the containing sphere. We construct the desired squared distance $\rho^2 + \rho_1^2 - d_E^2$ as $\rho^2 - (d_E^2 - \rho_1^2)$. The second term is the squared length of the tangent to sphere 1 from the center of sphere 2. Call the tangent point t; the total squared length is then found as the distance to sphere 2 measured perpendicularly to the tangent, from t (i.e., by extending the line from the center of sphere 1 through the tangent point t till it intersects sphere 2).

Chapter 15

Constructions in Euclidean Geometry

15.8 Exercises

15.8.1 Drills

1. Compute the tangent at the origin of the sphere Σ through the points at locations 0, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

   Answer: The tangent is a tangent bivector, given by the 3-blade $o \wedge \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_3)$

2. Factorize the circle K through the points at locations $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

   Answer: We compute, using GAViewer:

   $$K = \mathbf{e}_{123} + (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge o + \tfrac{1}{2}(\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge \infty.$$

   Then K's carrier is $K \wedge \infty = \mathbf{e}_{123} \wedge \infty + (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge o \wedge \infty$, and its surround is the dual sphere $\frac{K}{K \wedge \infty} = o + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/3 - \infty/6 = o + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/3 + \frac{1}{2}(1/3)\infty - \frac{1}{2}(2/3)\infty) = c - \frac{1}{2}(2/3)\infty$ (defining c and showing that the radius equals $\sqrt{2}/\sqrt{3}$). In the factored form, we should be able to write the carrier as a plane passing through the center c of the sphere. And indeed, $\mathbf{e}_{123} \wedge \infty + (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge o \wedge \infty = c \wedge (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge \infty$. In total, the factored form is:

   $$K = \big(c - \tfrac{1}{2}(2/3)\infty\big)\,\big(c \wedge (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge \infty\big),$$

   showing the K is the meet of a sphere and a plane.

3. Use that factorization of the circle K to spot its squared radius, center, carrier and surround, by inspection.

   Answer: The factorization is:

   $$K = \big(c - \tfrac{1}{2}(2/3)\infty\big)\,\big(c \wedge (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge \infty\big),$$

   with c the point at $1/3(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. The carrier is therefore $c \wedge (\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \wedge \infty\big)$, the surround the dual sphere $c - \frac{1}{2}(2/3)\infty\big)$, and the squared radius is $2/3$.

4. Project the point at the origin onto the carrier plane of the circle K.

   Answer: We project the point o at the origin onto the carrier plane Π given in the previous question through the formula $(o \rfloor \Pi)/\Pi = o + (e1 + e2 + e3)/3 + 1/3\infty$.

5. Make the free vector, tangent vector, line vector and normal vector in the direction $\mathbf{e}_1$, at the origin (if a location is required).

   Answer: Using the formulas in the book gives

   - Free vector: $\mathbf{e}_1 \wedge \infty$
   - Tangent vector: $o \wedge \mathbf{e}_1$
   - Line vector: $o \wedge \mathbf{e}_1 \wedge \infty$
   - Normal vector: $\mathbf{e}_1$

6. Rotate each of the vectors of the previous exercise by $\pi/2$ in the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane. Explain the results.

   Answer: Things rotate.

   - Free vector: $\mathbf{e}_2 \wedge \infty$
   - Tangent vector: $o \wedge \mathbf{e}_2$
   - Line vector: $o \wedge \mathbf{e}_2 \wedge \infty$
   - Normal vector: $\mathbf{e}_2$

7. Translate each of the vectors of the previous exercise by $\mathbf{e}_1 + \mathbf{e}_2$. Explain the results.

   Answer:

   - Free vector: $\mathbf{e}_2 \wedge \infty$ — Free vectors are unchanged by translations
   - Tangent vector: $(o + \mathbf{e}_1 + \mathbf{e}_2 + \infty) \wedge (\mathbf{e}_2 + \infty)$ — Don't forget to translate the Euclidean direction part
   - Line vector: $(o \wedge \mathbf{e}_2 + (\mathbf{e}_1 \wedge \mathbf{e}_2)) \wedge \infty$
   - Normal vector: $\mathbf{e}_2 + \infty$

15.8.2 Structural Exercises

1. Express the scalar product of two blades in terms of the scalar product of their duals. It should only differ by a sign, which you should express in terms of the grade of the blades and the space they reside in.

   Answer: Let $\mathbf{I}$ be the pseudoscalar of the space in which the dual is computed (at least as large as the join of $\mathbf{A}$ and $\mathbf{B}$).

$$\begin{aligned} \mathbf{A} * \mathbf{B} &= \langle \mathbf{A}\,\mathbf{B}\rangle_0 = \langle \mathbf{A}\,\mathbf{I}^{-1}\,\mathbf{I}\,\mathbf{B}\rangle_0 = \langle\ (\mathbf{A}\,\mathbf{I}^{-1})\,((-1)^{b(i+1)}\,\mathbf{B}\,\mathbf{I})\ \rangle_0 \\ &= (-1)^{b(i+1)+\frac{1}{2}i(i-1)}\langle\ \mathbf{A}^*\,\mathbf{B}^*\rangle_0 = (-1)^{(b+i/2)(i-1)}\,\mathbf{A}^* * \mathbf{B}^*, \end{aligned}$$

   with $b \equiv \text{grade}(\mathbf{B})$ and $i \equiv \text{grade}(\mathbf{I})$.

3. Show that the tangent of a tangent is zero. (Hint: Realize that a tangent is also a round; now use (15.1).)

   Answer: Since a tangent is just a regular element of the algebra we can indeed apply (15.1), and the result follows as $p\rfloor(p\rfloor\widehat{X}) = (p \wedge p)\rfloor\widehat{X} = 0$.

5. Show that the dual sphere $s = r\rfloor(p \wedge q) = (r\rfloor p)\,q - (r\rfloor q)\,p$ is a member of the parametrized family (15.6), passing through r (but non-normalized).

   Answer: The dual sphere s clearly passes through r, since $r \cdot s = 0$. If the point is non-normalized, we have an extra weight α, so we can set $\alpha(\lambda p + (1-\lambda)q) = -(r\cdot q)p + (r\cdot p)q$, and solve that $\alpha = r\cdot(p-q)$, $\lambda = 1/(1 - r\cdot p/r\cdot q)$.

7. Give the formula for the circle through a point pair $p \wedge q$ intersecting the dual plane π perpendicularly. Also, the circle having a tangent vector in direction $\mathbf{u}$ at p, and plunging into π.

   Answer: These are simple applications of the basic construction procedures: $p \wedge q \wedge \pi$ and $p \rfloor (p \wedge \infty \wedge \mathbf{u}) \wedge \pi$.

9. Construct the 'contour' of a sphere Σ as seen from a point p, i.e., the circle K of points where the invisible part of the sphere borders the visible part, as in Figure 15.13. (Hint: the white sphere in the figure is a clue to the construction. Express it first, using the plunge. Then construct the circle as a meet.

   Answer: The construction is based on the idea that a sphere Σ_p through that circle, with p at its center, plunges into Σ perpendicularly. So in dual form (using lower case for the duals), $s_p = s\rfloor(\textit{something})$. We also know that p is the center of s_p, which means it should plunge into the flat point $p \wedge \infty$ (as in Fig. 15.4b). So $s_p = s\rfloor(p \wedge \infty)$. Note that this generalizes (14.4), since s is now a general dual sphere, not merely a point. Then the circle we are looking for is obtained by the meet of S_p with S, which is done as

   $$K = \Big(\sigma \wedge \big(\sigma\rfloor(p \wedge \infty)\big)\Big)^{-*} = \sigma\rfloor\big(\sigma \wedge (p \wedge \infty)^{-*}\big).$$

   This is a pleasantly coordinate-free parameterization of the sought-for object. The advantage of such parameterizations appears when using the full geometric calculus of Chapter 8, in which we can directly differentiate such expressions to their constituents in a coordinate-free manner.

11. Show that the tangent with direction element $\mathbf{E}$ at p can be written in two equivalent forms:

   $$\textit{tangent } \mathbf{E} \textit{ at } p\textit{:} \quad p \wedge \big(-p\rfloor(\widehat{\mathbf{E}}\infty)\big) = p\rfloor(p \wedge \widehat{\mathbf{E}} \wedge \infty).$$

   Answer: Easy expansion of the contraction on both sides.

Chapter 16

Conformal Operators

16.10 Exercises

16.10.1 Drills

1. Reflect the line L through locations $\mathbf{e}_1$ and $\mathbf{e}_2$ in the unit sphere at the origin.

   Answer: $L = (o + \mathbf{e}_1) \wedge (o + \mathbf{e}_2) \wedge \infty = (o + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge \infty$. Then use pg. 466 to reflect L term by term, giving $C = (\infty/2 + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge (2o)$. This should be a circle, but that is not obvious.

2. Factorize the result of the previous exercise to determine its center and squared radius.

   Answer: Using (15.4), we compute the carrier $C \wedge \infty = 2\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge o \wedge \infty$. Then compute $S = C/(C \wedge \infty) = o + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$, which is a sphere with center at $(\mathbf{e}_1 + \mathbf{e}_2)/2$ and radius squared $S^2 = \rho^2$. The factorization cuts that sphere perpendicularly by the plane to produce the circle, of which center and radius are therefore $(\mathbf{e}_1 + \mathbf{e}_2)/2$ and $\pm\rho$. We recommend you to visualize these computations by GAviewer.

3. Reflect the tangent vector at $\mathbf{e}_1 + \mathbf{e}_2$ in the direction $2\mathbf{e}_3$ in the unit sphere at the origin. Notice especially the weight of the result!

   Answer: Set $p = o + \mathbf{e}_1 + \mathbf{e}_2 + \infty$, then the tangent vector is $T = p \rfloor (p \wedge (2\mathbf{e}_3) \wedge \infty) = -2(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3 + 2\mathbf{e}_3 \wedge (o + \infty)$. Reflecting using page 466 gives $T' = -2(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3 + 2\mathbf{e}_3 \wedge (2o + \infty/2)$. This is another tangent vector, interpretation can be done by factorization. Its carrier is $T' \wedge \infty = (o + (\mathbf{e}_1 + \mathbf{e}_2)/2) \wedge (-4\mathbf{e}_3) \wedge \infty$, showing that the weight is 4. Then $T'/(T' \wedge \infty)$ is its location, at $(\mathbf{e}_1 + \mathbf{e}_2)/2$.

4. Scale the line L by a factor of e^2, from the origin.

   Answer: $L = (o + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge \infty$, and using page 470 with $\gamma = 2$, gives $L' = (e^{-2}o + \mathbf{e}_1) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge (e^2\infty) = (o + (e^2\mathbf{e}_1)) \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge \infty$. So the line moves e^2 further out of the origin.

5. Scale the line L by a factor of e^2, from the point $\mathbf{e}_1$.

   Answer: Two methods: use the translated scaling versor $T_{\mathbf{e}_1} S T_{-\mathbf{e}_1}$, or translate the line to the origin, scale, and put back. It is then basically a scaling of $o \wedge (\mathbf{e}_2 - \mathbf{e}_1) \wedge \infty$, which is the identity. So L remains unchanged (see also structural exercise 4).

6. Reflect the line L in the origin.

   Answer: According to Section 16.3.2, this uses the versor $R = o \wedge \infty$. Be careful, its inverse is not its reverse, but $R^{-1} = R$. This has no effect on a Euclidean element, but $(o\wedge\infty)o(o\wedge\infty) = -o$ and $(o\wedge\infty)\infty(o\wedge\infty) = -\infty$, so the line $L = (o+\mathbf{e}_1)\wedge(\mathbf{e}_2-\mathbf{e}_1)\wedge\infty$ becomes $L' = (-o+\mathbf{e}_1)\wedge(\mathbf{e}_2-\mathbf{e}_1)\wedge(-\infty) = (o-\mathbf{e}_1)\wedge(\mathbf{e}_2-\mathbf{e}_1)\wedge\infty$. This is a line in the *same* direction, passing through the reflected location.

   You may have expected a line in the opposite direction; but the reflection of a point affects both its location *and its weight*. This applies also to the point at infinity, and that causes the line to seemingly preserve its direction. (Actually, the direction changes relative to the new support vector.)

   To study this strange effect more purely, note what the reflection $o\wedge\infty$ does on a tangent $o\,\mathbf{E}$: it becomes $(o\wedge\infty)(o\,\mathbf{E})(o\wedge\infty) = (o\wedge\infty)o(o\wedge\infty)\,\mathbf{E} = o(-\mathbf{E})$, precisely what you would expect. But this effect is canceled for a line by the sign change of ∞ under reflection in the origin.

7. Reflect the line L in the point $\mathbf{e}_1$.

   Answer: Now you should use the versor $T_{\mathbf{e}_1}(o\wedge\infty)T_{-\mathbf{e}_1} = (o+\mathbf{e}_1)\wedge\infty$. This transforms o into $-(o+2\mathbf{e}_1+2\infty)$, it transforms ∞ into $-\infty$ and the Euclidean vector $\mathbf{e}_1$ into $\mathbf{e}_1+2\infty$, and $\mathbf{e}_2$ into $\mathbf{e}_2$. Putting it together on $L = (o+\mathbf{e}_1)\wedge(\mathbf{e}_2-\mathbf{e}_1)\wedge\infty$ shows that the line is invariant, even after the previous exercise still a surprise.

16.10.2 Structural Exercises

1. Show that the general inversion formula for a point in a dual sphere at the origin $o - \frac{1}{2}\rho^2\infty$ is:

   $$\mathsf{T}_{\mathbf{x}}[o] \quad\mapsto\quad \frac{\mathbf{x}^2}{\rho^2}\,\mathsf{T}_{\rho^2\mathbf{x}^{-1}}[o].$$

   Note that this implies that imaginary spheres involve a central reflection in the origin (as well as the inversion of the distances to the origin).

   Answer: Compute complete analogous to the results on page 406. Do not forget that the inverse dual sphere is divided by its squared norm ρ^2.

   $$\begin{array}{ccl} o & \mapsto & -(o-\frac{1}{2}\rho^2\infty)\,o\,(o-\frac{1}{2}\rho^2\infty)/\rho^2 = -\frac{1}{4}\rho^2\infty\,o\,\infty = \infty\,\rho^2/2 \\ \infty & \mapsto & -(o-\frac{1}{2}\rho^2\infty)\,\infty\,(o-\frac{1}{2}\rho^2\infty)/\rho^2 = -o\,\infty\,o/\rho^2 = 2\,o/\rho^2 \\ \mathbf{E} & \mapsto & -(o-\frac{1}{2}\rho^2\infty)\,\mathbf{x}\,(o-\frac{1}{2}\rho^2\infty)/\rho^2 = \frac{1}{2}o\,\mathbf{x}\,\infty - \frac{1}{2}\infty\,\mathbf{x}\,o = \mathbf{x}. \end{array}$$

   Therefore the point $x = o + \mathbf{x} + \frac{1}{2}\mathbf{x}^2\infty$ transforms to $\infty\,\rho^2/2 + \mathbf{x} + \mathbf{x}^2/\rho^2\,o = (\mathbf{x}^2/\rho^2)\left(o + \rho^2\,\mathbf{x}^{-1} + \frac{1}{2}(\rho^2\mathbf{x}^{-1})^2\,\infty\right)$, which is the representation of the point at the location $\rho^2\,\mathbf{x}^{-1}$, with a weight of $\mathbf{x}^2/\rho^2$.

3. Figure 16.11 shows the reflection of various elements in the brown point pair; all elements reside in the plane of the drawing. Compare this to the spherical reflection Figure 16.1 and understand the differences. (Hint: Use the factorization of the point pair by a sphere and well chosen planes.)

   Answer: The hint helps you to see this as a multi-step operator; clearly one of the planes should be chosen to be the drawing plane.

5. We claimed in Section 16.4 that a transversion can also be constructed as the reflection in two touching spheres. To determine the standard form of such a transversion, put the spheres symmetrically around the origin with their origin at $\pm\mathbf{a}$. Show that this gives the transversion versor as $-\mathbf{a}^2\,(1 - 2o\wedge\mathbf{a}^{-1})$.

What should you take as the distance of the touching spheres to obtain the standard transversion rotor $\exp(-o \wedge \mathbf{t}/2)$?

Answer: We get $o - \mathbf{a})\,(o + \mathbf{a}) = -\mathbf{a}^2 + 2o \wedge \mathbf{a} = -\mathbf{a}^2(1 - 2o \wedge \mathbf{a}^{-1} = -\mathbf{a}^2 \exp(-2o\,\mathbf{a}^{-1})$. So, barring a scaling factor, we should set $\mathbf{a} = 4/\mathbf{t}$.

7. In Figure 16.12, we have generated a torus by generating a few circles. the inversion in a sphere of a torus is called *Dupin's cycloid*. In the conformal model, its circles are simply the torus circles defining the torus inverted in the sphere. Write pseudocode to generate this figure with just a few parametrized operations.

   Answer: You can look at the GAviewer code for this figure. The principles behind it are simple: generate the green torus as a collection of circles, and reflect each of those in the dual sphere (using it as a versor) to generate the red circles describing Dupin's cycloid.

   Let us take as 'wheel axis' for the torus the line $L = o \wedge \mathbf{e}_2 \wedge \infty$. Take a standard point $p = \rho_1 \mathbf{e}_1$ at the 'inner tube radius'. A direct circle through that point around the origin is $C = (p + \rho_1^2 \infty) \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$. Translate that using the versor $\exp(-\rho_2 \mathbf{e}_1 \infty)$ to get it at the proper 'wheel radius', and rotate in k steps around the line L using the versor $\exp(L^* \pi/k)$ repeatedly. That gives the direct representation of the rings around the tube.

   To generate the rings around the wheel rim, take the standard point p and rotate it in the $\mathbf{e}_1 \wedge \mathbf{e}_3$ plane over the desired number of steps to generate points along the tube. Translate all results over $\rho_2 \mathbf{e}_1$ to place them at the wheel rim. Now use those translated points p_i to generate circles along the wheel, directly, as $p_i \wedge L^*$.

9. Figure 16.13 depicts the same situation, at the same scale, of a green line $L = p \wedge q \wedge i$ through two points p and q (one of which is the center of the black circles), and a flat point $r \wedge i$ (in blue). In all three figures, the line is used in a translation versor $\exp(i \rfloor L/2)$ to translate a the point p over equal distances, and dual circles are made with the original point as its center as $t \rfloor (p \wedge i)$. The only difference is the element used for the infinity i (in light red). Identify the metrics and explain the differences, as quantitatively as you can. You can pick i from $\mathbf{e}_1$, o, ∞, $o - \infty/2$, $o + \infty/2$, $o + \mathbf{e}_2 + \infty/2$.

   Answer: A better feeling for these figures may be obtained by studying them interactively using GAviewer (type `FIG(16,13)`). Note that $o + \mathbf{e}_2 + \infty/2$ is a point. (a) is a hyperbolic geometry, so $i = o + \infty/2$; (b) is a Euclidean geometry, so $i = \infty$; (c) has a plane as its infinity, $i = \mathbf{e}_1$; (f) is a spherical geometry, $i = o + \infty/2$. That leaves $i = o$ and $i = o + \mathbf{e}_2 + \infty/2$, both points. Due to the translation invariance of the conformal geometries, these should really lead to the same phenomena. Only by knowing more about the locations could you truly tell them apart. In fact, the former is (d), the latter (e). Both can be seen as Euclidean planar geometry inverted in a circle (since that turns the Euclidean invariant ∞ into a point at the center of the circle).

Chapter 18

Implementation Issues

18.4 Structural Exercises

1. In a 2D Euclidean geometric algebra on an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, the general element X can be written as:

$$X = x_0 + x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_{12}\,\mathbf{e}_{12}.$$

According to Table 18.1, this algebra should be representable as the matrix algebra $\mathbb{R}(2)$, i.e., we should be able to find a 2×2 matrix representing the general element so that the geometric product of elements is represented as the matrix product. Show that the following works:

$$[\![X]\!] = \left[\!\!\left[\begin{array}{cc} x_0 + x_2 & x_1 - x_{12} \\ x_1 + x_{12} & x_0 - x_2 \end{array}\right]\!\!\right]$$

(This is not unique, some permutations of the same principles work as well, but not all. Why must the scalar part always be on the diagonal?)

Answer: Since matrix multiplication is linear and associative (and so is the geometric product), the simplest way to check this is to make the representation of the basis elements $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$ and verify that their geometric products are correct. For instance, $[\![\mathbf{e}_1]\!] = \left[\!\!\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\!\!\right]$ and $[\![\mathbf{e}_2]\!] = \left[\!\!\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\!\!\right]$, and multiplying we find $\left[\!\!\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]\!\!\right]$, which is indeed $[\![\mathbf{e}_{12}]\!]$. Check all of them in this way.

3. For a 3D Euclidean vector space, the matrix representation $\mathbb{C}(2)$ from Table 18.1 may be generated by:

$$[\![X]\!] = \left[\!\!\left[\begin{array}{cc} z & x + iy \\ x - iy & -z \end{array}\right]\!\!\right],$$

where i is the complex imaginary (so $i^2 = -1$). Verify this and compute the representation of a general element of this algebra.

Answer: You may realize that the complex numbers can of course be viewed as 2-D rotation operators and therefore can be represented by 2×2 matrices, with $[\![1]\!] = \left[\!\!\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\!\!\right]$ and $[\![i]\!] = \left[\!\!\left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]\!\!\right]$. Then this is exactly like like the previous exercise, and the result is

$$\left[\!\!\left[\begin{array}{cc} (\alpha_0 + \alpha_3) + i(-\alpha_{12} - \alpha_{123}) & (\alpha_1 + \alpha_{31}) + i(\alpha_2 - \alpha_{23}) \\ (\alpha_1 - \alpha_{31}) + i(-\alpha_2 - \alpha_{23}) & (\alpha_0 - \alpha_3) + i(\alpha_{12} - \alpha_{123}) \end{array}\right]\!\!\right].$$

The complex numbers group some of the inherent symmetries of the geometric algebra structure.

Chapter 20

The Linear Products and Operations

20.3 Structural Exercises

1. Why are the first columns of $[\![A^G]\!]$ and $[\![A^O]\!]$ equal to $[\![A]\!]$?

   Answer: Essentially, because $A\,1 = A \wedge 1 = A$. And because $A \rfloor 1 = \langle A \rangle_0$, the first column of $[\![A^L]\!]$ only contains the scalar component $\langle A \rangle_0$.

3. You know that $\mathbf{a}\,\mathbf{B} = \mathbf{a} \rfloor \mathbf{B} + \mathbf{a} \wedge \mathbf{B}$ for a vector $\mathbf{a}$ and a blade $\mathbf{B}$. How do you recognize this fact in the relationship of $[\![A^G]\!]$, $[\![A^L]\!]$ and $[\![A^O]\!]$? Why do we not have $[\![A^G]\!] = [\![A^L]\!] + [\![A^O]\!]$?

   Answer: Because the additive property only holds for general vectors. So certainly if A is a vector that has only A_1, A_2 and A_3 non-zero, this holds. Indeed at those entries of $[\![A^G]\!]$ that consist of $\pm A_1$, $\pm A_2$ and $\pm A_3$, it is the sum of the corresponding entries from the other matrices as well. There are other entries that sum as well, for instance the one corresponding to $(A_{12}\mathbf{e}_{12})\,\mathbf{e}_3 = (A_{12}\mathbf{e}_{12}) \rfloor \mathbf{e}_3 + (A_{12}\mathbf{e}_{12}) \wedge \mathbf{e}_3 = 0 + A_{12}\mathbf{e}_{123}$ (i.e., entry $(4, 8)$).

Chapter 22

Specializing the Structure for Efficiency

22.7 Structural Exercises

1. The weighted sum of which basis blades is required to represent a plane in general position on the o,∞-basis? And on the e, $\bar{e}$-basis?

   Answer: Let us answer this for 3-D. Then a plane is a 4-blade of the form $p \wedge \mathbf{A} \wedge \infty$, with $\mathbf{A}$ a Euclidean bivector. Working it out, all that remains of p are the non-∞ parts, so a 4-dimensional basis of $o \wedge \mathbf{e}_{ij} \wedge \infty$ and $\mathbf{e}_{ijk} \wedge \infty$ is sufficient. When using e and $\bar{e}$, by virtue of $o \wedge \infty = e \wedge \bar{e}$ the first part of this is the same, but making ∞ requires both e and $\bar{e}$, so 1 more basis element is required.

3. Is there a difference in the basis for rotors and versors?

   Answer: Rotors are essentially properly scaled versors. In a versor one could always set the first component to 1 and save a coordinate (at least if you are always planning to use it in a versor product), but in principle they require the same basis.